

EMBEDDINGS OF DECOMPOSITION SPACES

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ABSTRACT. A unifying framework for a wide class of smoothness spaces used in harmonic analysis— (α) -modulation spaces, Besov spaces and a large class of wavelet-type coorbit spaces—is provided by the theory of *decomposition spaces*. Put broadly, in this paper we ask (and answer) the following question: Given two decomposition spaces, is there an embedding between the two?

A decomposition space $\mathcal{D}(\mathcal{Q}, L^p, Y)$ can be fully described using only three ingredients: a covering $\mathcal{Q} = (Q_i)_{i \in I}$ of the frequency domain, an integrability exponent $p \in (0, \infty]$ and a sequence space $Y \leq \mathbb{C}^I$ on the index set I . Given these ingredients, the decomposition space norm of a function/distribution g is calculated by decomposing the frequency content of g into different parts according to the covering \mathcal{Q} . Each “frequency *localized* piece” is measured in the L^p norm and the contributions of the individual parts are *globally* aggregated using the sequence space Y . More formally, we have $\|g\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} = \left\| \left(\|\mathcal{F}^{-1}(\varphi_i \widehat{g})\|_{L^p} \right)_{i \in I} \right\|_Y$, where $(\varphi_i)_{i \in I}$ is a suitable partition of unity subordinate to \mathcal{Q} .

Up to now, the theory of decomposition spaces was mainly used to simplify the introduction of new spaces, e.g. to justify independence of the resulting space from the chosen partition of unity $(\varphi_i)_{i \in I}$. In this paper, we will show that identification of a given function space X as a decomposition space offers much greater advantages: Once this is done, assertions about (non)-existence of embeddings $X \hookrightarrow V$ or $V \hookrightarrow X$, for other decomposition spaces V , come essentially for free.

More precisely, we establish readily verifiable criteria which ensure existence of an embedding $\mathcal{D}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}(\mathcal{P}, L^{p_2}, Z)$, mostly concentrating on the case of weighted Lebesgue spaces, i.e., $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$. Since in this case, the two decomposition spaces depend only on the quantities $\mathcal{Q}, \mathcal{P}, w, v$ and p_1, p_2, q_1, q_2 , it should be possible to decide existence of the embedding only in terms of these ingredients. It is not at all clear, however, precisely *which* conditions on these quantities ensure/prevent existence of such an embedding. As we will see—under suitable assumptions on \mathcal{Q}, \mathcal{P} —the relevant sufficient conditions are $p_1 \leq p_2$, combined with finiteness of a nested norm of the form

$$\left\| \left(\left\| (\alpha_i \cdot v_j / w_i)_{i \in I_j} \right\|_{\ell^t} \right)_{j \in J} \right\|_{\ell^s} < \infty,$$

where the sets

$$I_j = \{i \in I \mid Q_i \cap P_j \neq \emptyset\} \quad \text{for } j \in J$$

are defined only in terms of the geometry of the two coverings $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$. Likewise, the exponents t, s and the weight α also depend only on the quantities mentioned above.

Thus, in spite of the Fourier analytic nature of decomposition spaces, *no knowledge of Fourier analysis is required for the application of our criteria*, since these are purely combinatorial. Finally, we show – under suitable assumptions on the two coverings \mathcal{Q}, \mathcal{P} – that the developed criteria are *sharp*. Precisely, for almost arbitrary coverings and certain ranges of p_1, p_2 , our criteria yield a *complete characterization* of the existence of an embedding. The same holds for *arbitrary* values of p_1, p_2 under more stringent assumptions on the two coverings \mathcal{Q}, \mathcal{P} .

As a further application of our proof techniques, we show a *rigidity result*, namely that—for $(p_1, q_1) \neq (2, 2)$ —two decomposition spaces $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$ and $\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$ can only *coincide* if their “ingredients” are equivalent, i.e. if $p_1 = p_2$ and $q_1 = q_2$ and if the coverings \mathcal{Q}, \mathcal{P} and the weights w, v are equivalent in a suitable sense.

We illustrate the power and convenience of the resulting embedding theory for decomposition spaces by applications to α -modulation spaces and Besov spaces. We will see that all known embedding results for these spaces are special cases of our general approach; in some cases, our criteria even show that the previously known embedding results can be improved considerably. Due to the length of the present paper, we postpone further applications—e.g. to shearlet smoothness spaces and wavelet-type coorbit spaces, including shearlet coorbit spaces—to a later contribution.

Finally, it is worth pointing out that our criteria allow for $p_1 \neq p_2$, so that one can “trade smoothness for integrability” in a certain sense, similar to the classical Sobolev embedding theorems. For $\mathcal{Q} \neq \mathcal{P}$, sharp embeddings of this generality have previously not been known, even for the comparatively well-studied special case of α -modulation spaces.

Key words and phrases. Smoothness spaces, Decomposition spaces, Embeddings, Besov spaces, α -modulation spaces, Coorbit spaces.

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1. INTRODUCTION

Our aim in this paper is to establish a general and easy-to-use framework for proving and disproving the existence of embeddings between decomposition spaces.

The structure of this introduction is as follows: First, we illustrate the relevance of a general framework for embeddings between decomposition spaces by discussing related work. As we will see, the existing literature shows that there is quite some interest in embeddings between different special classes of decomposition spaces. All of these results can be obtained with ease using the framework presented in this paper. Furthermore, recent work in the theory of (wavelet type) coorbit spaces shows that certain decomposition spaces defined using “exotic, non-classical” coverings are of interest. This further increases the need for a general, comprehensive framework for such embeddings.

Next, we fix the notations used in the remainder of the paper. Most of these notational conventions are standard and are only given here to fix e.g. the exact normalization of the Fourier transform which we use. There are, however, some non-standard notations which are crucial in the remainder of the paper, namely the **upper/lower (signed) conjugate exponents** p^Δ, p^∇ and $p^{\pm\Delta}$.

Having introduced our notation, we give a quick overview of the results obtained in this paper, including a very brief overview over the most important notions.

Finally, we close the introduction with an explanation of the general structure of the paper.

1.1. Motivation and related work. Let us start at the beginning: Decomposition spaces were first introduced—in a very general setting—by Feichtinger and Gröbner[7, 6], with the aim of giving a unifying framework for *Besov spaces*, *modulation spaces* and certain *Wiener amalgam spaces*.

Furthermore, Gröbner then used the framework of decomposition spaces in his PhD thesis[11] to define the so-called α -modulation spaces $M_{s,\alpha}^{p,q}(\mathbb{R}^d)$ for different values of $\alpha \in [0, 1]$. These spaces satisfy (up to canonical identifications)

$$M_{s,\alpha}^{p,q}(\mathbb{R}^d) = \mathcal{D}\left(\mathcal{Q}^{(\alpha)}, L^p, \ell_{u^{(s,\alpha)}}^q\right)$$

for a so-called α -covering $\mathcal{Q}^{(\alpha)}$ of \mathbb{R}^d and a suitable weight $u^{(s,\alpha)}$. The point of these coverings is that they “geometrically interpolate” the dyadic covering used to construct (inhomogeneous) Besov spaces and the uniform covering with which modulation spaces are defined. Thus, α -modulation spaces can be considered as intermediate spaces between modulation- and Besov spaces. Note, however, that they can *not* be obtained[12] by complex interpolation between those spaces.

In his thesis, Gröbner achieved certain sufficient and certain necessary conditions for existence of the embedding

$$M_{s_1,\alpha_1}^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M_{s_2,\alpha_2}^{p_2,q_2}(\mathbb{R}^d), \quad (1.1)$$

but did not obtain a complete characterization. More recently, Sugimoto and Tomita[17] characterized the embeddings between modulation spaces and Besov spaces, which amounts to the above embedding for $(\alpha_1, \alpha_2) \in \{(0, 1), (1, 0)\}$. For general α_1, α_2 , the work of Gröbner was continued by Toft and Wahlberg[19], shortly before existence of the embedding (1.1) was characterized completely—at least for $(p_1, q_1) = (p_2, q_2)$ —by Han and Wang[13]. Some of their techniques were used in my PhD thesis[23] on which large parts of this paper are based. As we will see in the last part of this paper, the present results greatly generalize those of Han and Wang, since they characterize the embedding (1.1) *completely*, also for $(p_1, q_1) \neq (p_2, q_2)$.

Borup and Nielsen also contributed heavily to the theory of decomposition spaces. In [4], they introduced a more restrictive class of decomposition spaces than those originally defined by Feichtinger and Gröbner. It is (essentially) this class of decomposition spaces that we will consider in this paper. Furthermore, they introduced the class of structured coverings and gave a general construction of Banach frames for decomposition spaces, which they first formulated[2] in the context of α -modulation spaces. Additionally, they studied boundedness properties of certain pseudo-differential operators on α -modulation spaces[3].

A further application of decomposition spaces is the paper [15] in which Labate et al. define so-called *shearlet smoothness spaces* and study (among other things) the existence of embeddings between these spaces and (inhomogeneous) Besov spaces. Note, however, that their definition of Besov spaces does not coincide with the usual one, so that their results need to be taken with a grain of salt. Using the results in this paper, the existence of the embeddings studied in [15] can be *characterized completely* (cf. [23, Theorem 6.4.3]), whereas Labate et al. only give sufficient criteria.

We would also like to mention the recent paper [26], in which boundedness properties of certain Fourier multipliers on α -modulation spaces are studied.

Finally, we mention the work [9] of Hartmut Führ and the present author, in which it is shown that a large class of (generalized) wavelet-type coorbit spaces are canonically isomorphic to certain decomposition spaces. Precisely, if $H \leq \text{GL}(\mathbb{R}^d)$ is an *admissible dilation group*, we have (up to canonical identifications) that

$$\text{Co}(L_m^{p,q}(\mathbb{R}^d \rtimes H)) = \mathcal{D}(\mathcal{Q}_H, L^p, \ell_u^q), \quad (1.2)$$

where \mathcal{Q}_H is a so-called **induced covering** of the **dual orbit** $\mathcal{O} = H^T \xi_0$ of H . Here, $\xi_0 \in \mathbb{R}^d$ is chosen such that \mathcal{O} is open and of full measure; existence of such ξ_0 is part of the definition of an admissible dilation group. Originally, this result was only proven in the Banach setting $p, q \in [1, \infty]$, but in my PhD thesis[23], this was extended to the full range $p, q \in (0, \infty]$.

Using the isomorphism (1.2), one can deduce many properties of the coorbit space $\text{Co}(L_m^{p,q}(\mathbb{R}^d \rtimes H))$ which are not obvious from the coorbit-space point of view. In particular, combined with the embedding results in this paper, one can obtain nontrivial embeddings between wavelet-type coorbit spaces with respect to different dilation groups and also embeddings between these coorbit spaces and classical function spaces like Besov spaces. Using the results from [24], one can also obtain embeddings

into the classical Sobolev spaces $W^{k,q}(\mathbb{R}^d)$. For some results in this direction, see [23, Sections 6.3 and 6.5] and [24, Section 7]. Due to the considerable length of the present paper, we postpone a more detailed discussion of these examples to a later contribution.

1.2. Notation. We let $\mathbb{N} := \{n \in \mathbb{Z} \mid n \geq 1\}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. For $n \in \mathbb{N}_0$, we write $\underline{n} := \{1, \dots, n\}$. In particular, $\underline{0} = \emptyset$. For $x \in \mathbb{R}$, we denote the positive part of x by $x_+ := \max\{x, 0\}$.

We write $|x|$ for the euclidean norm of $x \in \mathbb{R}^d$. The *open* euclidean ball of radius $r > 0$ around $x \in \mathbb{R}^d$ is denoted by $B_r(x)$ and the corresponding closed ball is denoted by $\overline{B}_r(x)$. For subsets $A, B \subset \mathbb{R}^d$, we write $A - B$ for the **Minkowski difference** (or **algebraic difference**)

$$A - B = \{a - b \mid a \in A, b \in B\},$$

which should not be confused with the set-theoretic difference $A \setminus B = \{a \in A \mid a \notin B\}$.

For a subset $M \subset V$ of a vector space V , we write $\langle M \rangle$ for the span of M . If $(x_i)_{i \in I}$ is a family in V , we also write $\langle x_i \mid i \in I \rangle$ for $\langle \{x_i \mid i \in I\} \rangle$.

A **quasi-norm** $\|\cdot\|$ on a vector space X has to satisfy the same properties as a norm, with the exception that the triangle inequality is replaced by the **quasi-triangle inequality**

$$\|x + y\| \leq C \cdot (\|x\| + \|y\|) \quad \forall x, y \in X,$$

where the **triangle constant** $C \geq 1$ of X is independent of $x, y \in X$. Cauchy sequences, convergence of sequences and completeness is defined as for normed vector spaces. A complete quasi-normed space is called a **Quasi-Banach space**. The two main differences between normed vector spaces and quasi-normed vector spaces are that a quasi-norm $\|\cdot\|$ need not be continuous and that the topological dual X' of a quasi-normed space need not separate the points of X .

For the interior of a subset M of a topological space X , we write M° . The closure of M is denoted by \overline{M} . Given an open subset $U \subset M$, we write $C_c(U)$ for the space of all continuous functions $f : X \rightarrow \mathbb{C}$ which have compact support $\text{supp } f \subset U$.

For a function $f : G \rightarrow S$, for some group G and some set S , we write

$$\begin{aligned} L_x f : G &\rightarrow S, y \mapsto f(x^{-1}y), \\ R_x f : G &\rightarrow S, y \mapsto f(yx), \\ f^\vee : G &\rightarrow S, y \mapsto f(y^{-1}) \end{aligned}$$

for each $x \in G$. For $G = \mathbb{R}^d$, we occasionally also write $T_x := L_x$.

Furthermore, for $f : \mathbb{R}^d \rightarrow \mathbb{C}$ and $\xi \in \mathbb{R}^d$, we define

$$M_\xi f : \mathbb{R}^d \rightarrow \mathbb{C}, x \mapsto e^{2\pi i \langle x, \xi \rangle} \cdot f(x).$$

In the same setting, for $h \in \text{GL}(\mathbb{R}^d)$, we let

$$\Delta_h f := f \circ h^T.$$

In case of $h = a \cdot \text{id}$ for some $a \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, we also write $\Delta_a := \Delta_h$.

For the **Fourier transform**, we use the normalization

$$(\mathcal{F}f)(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) \cdot e^{-2\pi i \langle x, \xi \rangle} \, dx$$

for $f \in L^1(\mathbb{R}^d)$. As is well known, \mathcal{F} extends to a unitary automorphism of $L^2(\mathbb{R}^d)$, where the inverse is the unique extension of the operator \mathcal{F}^{-1} , given on $L^1(\mathbb{R}^d)$ by

$$(\mathcal{F}^{-1}g)(x) := \int_{\mathbb{R}^d} g(\xi) \cdot e^{2\pi i \langle x, \xi \rangle} \, d\xi.$$

The **Lebesgue measure** of a (measurable) subset $M \subset \mathbb{R}^d$ will be denoted by $\lambda(M)$. The **cardinality** of a set M (which will either be a nonnegative integer or ∞) is denoted by $|M|$.

If X is a set and $M \subset X$, we write $\chi_M := \mathbb{1}_M$ for the **characteristic function** (or **indicator function**)

$$\chi_M : X \rightarrow \{0, 1\}, x \mapsto \begin{cases} 0, & \text{if } x \notin M, \\ 1, & \text{if } x \in M. \end{cases}$$

Given two families $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ of sets, we define the **intersection sets** I_j and J_i for $i \in I$ and $j \in J$ by

$$I_j := \{\ell \in I \mid Q_\ell \cap P_j \neq \emptyset\} \quad \text{and} \quad J_i := \{\ell \in J \mid P_\ell \cap Q_i \neq \emptyset\}.$$

For $p \in [1, \infty]$, we let $p' \in [1, \infty]$ denote the usual **conjugate exponent** of p , i.e. such that $\frac{1}{p} + \frac{1}{p'} = 1$. For $p \in (0, 1)$, however, we define the conjugate exponent of p as $p' := \infty$. Note that $\frac{1}{p} + \frac{1}{p'} = 1$ is thus *not* fulfilled for $p \in (0, 1)$.

Furthermore, we define the **upper conjugate exponent** p^Δ of $p \in (0, \infty]$ by $p^\Delta = \max\{p, p'\}$, whereas the **lower conjugate exponent** is given by $p^\nabla := \min\{p, p'\}$. Finally, the **signed upper conjugate exponent** $p^{\pm\Delta}$ is defined by

$$p^{\pm\Delta} := \begin{cases} p, & \text{if } p \geq 2, \\ \frac{p}{p-1}, & \text{if } 0 < p < 2 \text{ and } p \neq 1, \\ \infty, & \text{if } p = 1, \end{cases}$$

which ensures that $1/p^{\pm\Delta} = \min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\}$.

Our notation and nomenclature for these exponents is designed to be mnemonic: p^Δ is the *upper* conjugate exponent, since we are taking the *maximum* of p and its conjugate p' . For the same reason, we choose an *upwards* pointing triangle as the exponent of p^Δ . Our conventions for p^∇ are explained similarly.

Finally, note that we have $1/p^\Delta = \min\left\{\frac{1}{p}, \frac{1}{p'}\right\}$, with $\frac{1}{p'} = 1 - \frac{1}{p}$ for $p \in [1, \infty]$, but $\frac{1}{p'} = 0$ for $p \in (0, 1)$. Thus, if we were to define p' such that $\frac{1}{p'} = 1 - \frac{1}{p}$ for all $p \in (0, \infty]$, we would arrive at $1/p^\Delta = \min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\}$. Note that this identity *does* hold (by definition) for $p^{\pm\Delta}$. Furthermore, note that this entails $1/p^{\pm\Delta} = 1 - \frac{1}{p} < 0$ for $p \in (0, 1)$, so that $p^{\pm\Delta}$ is signed—in contrast to $p^\Delta, p^\nabla \in (0, \infty]$. This explains the notation and nomenclature of the *signed* upper conjugate exponent $p^{\pm\Delta}$.

A comment on the writing style and the length of the paper. The reader might object that the length of this paper is excessive. But this length is due to two main factors:

- The paper has three different main aims which could in principle each be handled in a separate paper, namely we
 - establish *sufficient* conditions for the existence of embeddings,
 - establish *necessary* conditions for the existence of embeddings,
 - simplify these conditions to the point where they are practically usable.

But since these three aims are highly interwoven and dependent on each other, I decided to instead write one larger contribution.

- Compared with many other papers, the writing style of this paper is rather detailed. Furthermore, I try to explain some of the ideas which led to the development of the results, instead of just giving a formally correct but unenlightening proof. Thus, even though the *number of pages* might be larger than for many other papers, it is hoped that the required *reading time* is not as high as one would expect, simply because the reader does not have to work as hard.

1.3. Overview of new results. In this subsection, we try to give a flavor of the framework for embeddings between decomposition spaces which will be developed in this paper, by presenting the most important results. To do this, we first need to clarify the setting in which we work, by introducing a few central concepts.

1.3.1. Concepts related to decomposition spaces. Although the results of this paper apply in greater generality, in this introduction, we will restrict ourselves to the setting of **almost structured coverings**. Above all, such a covering $\mathcal{Q} = (Q_i)_{i \in I}$ of an open set $\mathcal{O} \subset \mathbb{R}^d$ is required to have the **finite overlap property**. If we let

$$i^* := \{\ell \in I \mid Q_\ell \cap Q_i \neq \emptyset\}$$

denote the **set of neighbors** of Q_i , the finite overlap property simply means that the number of neighbors of each set is uniformly bounded, i.e. that the following constant is finite:

$$N_Q := \sup_{i \in I} |i^*|.$$

A further property of almost structured coverings is that one can associate to $\mathcal{Q} = (Q_i)_{i \in I}$ a family of (invertible) affine transformations $(\xi \mapsto T_i \xi + b_i)_{i \in I}$ such that—essentially—every set Q_i is obtained as the affine image of a fixed set $Q \subset \mathbb{R}^d$. The actual definition allows for a slightly larger degree of freedom. Precisely, \mathcal{Q} is called an almost structured covering of the open subset \mathcal{O} of the frequency space \mathbb{R}^d , if there are *open, bounded* subsets $Q'_i \subset \mathbb{R}^d$ and invertible affine transformations as above such that the following hold:

- (1) \mathcal{Q} has the finite overlap property and $Q_i = T_i Q'_i + b_i$ for all $i \in I$.
- (2) The following constant is finite:

$$C_Q := \sup_{i \in I} \sup_{\ell \in i^*} \|T_i^{-1} T_\ell\|.$$

- (3) For each $i \in I$, there is an open set $P'_i \subset \mathbb{R}^d$ satisfying $\overline{P'_i} \subset Q'_i$ and the following conditions:
 - (a) The sets $\{P'_i \mid i \in I\}$ and $\{Q'_i \mid i \in I\}$ are finite.
 - (b) We have $\mathcal{O} = \bigcup_{i \in I} (T_i P'_i + b_i) = \bigcup_{i \in I} Q_i$.

Given such a covering, we define the associated **decomposition space (quasi)-norm** of a distribution¹ g as

$$\|g\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)} := \left\| \left(\|\mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})\|_{L^p} \right)_{i \in I} \right\|_Y, \quad (1.3)$$

for a suitable *partition of unity* $\Phi = (\varphi_i)_{i \in I}$ subordinate to \mathcal{Q} and a suitable *sequence space* $Y \leq \mathbb{C}^I$. Precisely, we assume that Φ is a so-called **L^p -BAPU**, a concept whose precise definition is immaterial for this introduction. It suffices to know that we have in particular $\varphi_i \in C_c^\infty(Q_i)$ and $\sum_{i \in I} \varphi_i \equiv 1$ on \mathcal{O} . Nevertheless, an important property of almost structured coverings is that there is a family $(\varphi_i)_{i \in I}$ which is an L^p -BAPU simultaneously for all $p \in (0, \infty]$, cf. Theorem 3.19.

Finally, to ensure that the (quasi)-norm in equation (1.3) is well-defined (i.e., independent of the chosen BAPU), we require Y to be **\mathcal{Q} -regular**. This means the following:

- (1) Y is a Quasi-Banach space and a subspace of the space \mathbb{C}^I of all sequences over I .
- (2) Y is **solid**, i.e., if $y = (y_i)_{i \in I} \in Y$ and $x = (x_i)_{i \in I} \in \mathbb{C}^I$ satisfy $|x_i| \leq |y_i|$ for all $i \in I$, then $x \in Y$ with $\|x\|_Y \leq \|y\|_Y$.
- (3) The **\mathcal{Q} -clustering map** $\Gamma_Q : Y \rightarrow Y, x \mapsto x^*$ with $x_i^* := \sum_{\ell \in i^*} x_\ell$ is well-defined and bounded.

As we will see, given these assumptions, the decomposition space (quasi)-norm (1.3) is well-defined and the associated decomposition space $\mathcal{D}(\mathcal{Q}, L^p, Y)$ is a Quasi-Banach space.

The most common type of \mathcal{Q} -regular sequence spaces that we will consider are the **weighted ℓ^q spaces**: If I is a set and if $w = (w_i)_{i \in I}$ is a weight, then

$$\ell_w^q(I) := \{x = (x_i)_{i \in I} \in \mathbb{C}^I \mid (w_i \cdot x_i)_{i \in I} \in \ell^q(I)\},$$

with the obvious (quasi)-norm. Of course, we have to make sure that $Y = \ell_w^q(I)$ is \mathcal{Q} -regular. This is true (cf. Lemma 4.13) if w is **\mathcal{Q} -moderate**, i.e., if the following constant is finite:

$$C_{w, \mathcal{Q}} := \sup_{i \in I} \sup_{\ell \in i^*} \frac{w_i}{w_\ell}.$$

We close this short overview of our general setting with a remark on the **Fourier-side decomposition space** $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$. The decomposition space $\mathcal{D}(\mathcal{Q}, L^p, Y)$ as defined by the (quasi)-norm in equation (1.3) has the advantage that known spaces like Besov spaces can be modeled easily as decomposition spaces. Indeed, if \mathcal{Q} is the dyadic covering and if the sequence space $Y = Y^{(q, s)}$ is chosen suitably, we have

$$\|g\|_{B_s^{p, q}} \asymp \|g\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)} \quad \forall g \in \mathcal{S}'(\mathbb{R}^d).$$

¹Strictly speaking, the decomposition space $\mathcal{D}(\mathcal{Q}, L^p, Y)$ will usually *not* be a space of (tempered) distributions. Instead, it will be a subspace of $\mathcal{F}^{-1}(\mathcal{D}'(\mathcal{O}))$, i.e. of the inverse image under the Fourier transform of the space of distributions on \mathcal{O} .

Nevertheless, this definition also has several disadvantages, which we will briefly discuss now: To this end, note that the expression $\mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})$ amounts to localizing the frequency-content of g to the set Q_i . If—instead of g itself—we were to consider the “Fourier-side” object $h := \widehat{g}$ as our primary object of study, then this frequency localization would simply be a matter of multiplying with φ_i , instead of going back and forth between the “space” and “Fourier” side. In this paper, we will need to consider such frequency localizations *extremely* often; thus, we will mainly consider the “Fourier side” versions of our functions under consideration. In doing this, we only need to take care to “go back to the space side” when we want to calculate L^p -norms.

Precisely, for a distribution $h \in \mathcal{D}'(\mathcal{O})$ on \mathcal{O} , we define (similar to equation (1.3)) the **Fourier-side decomposition space (quasi)-norm** of h as

$$\|h\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} := \left\| \left(\|\mathcal{F}^{-1}(\varphi_i \cdot h)\|_{L^p} \right)_{i \in I} \right\|_Y$$

and define the Fourier-side decomposition space as

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y) := \left\{ h \in \mathcal{D}'(\mathcal{O}) \mid \|h\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} < \infty \right\}.$$

As a further advantage, note that $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ is a subspace of the well-known space $\mathcal{D}'(\mathcal{O})$ of distributions on \mathcal{O} , while $\mathcal{D}(\mathcal{Q}, L^p, Y)$ —when defined correctly—is a subspace of the (slightly awkward) space $Z'(\mathcal{O}) := \mathcal{F}^{-1}(\mathcal{D}'(\mathcal{O}))$. This will be discussed in more detail in Subsection 3.2, see in particular Remark 3.13.

Finally, note that the Fourier transform $\mathcal{F} : \mathcal{D}(\mathcal{Q}, L^p, Y) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ is an isomorphism, so that it does not really matter which of the spaces we consider; all results (in particular embeddings) for the Fourier-side spaces easily translate to results about the space-side spaces and vice versa.

1.3.2. Sufficient conditions for embeddings. In the following, we always assume that we are given two almost structured coverings

$$\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I} \quad \text{and} \quad \mathcal{P} = (P_j)_{j \in J} = (S_j P'_j + c_j)_{j \in J}$$

of the open subsets $\mathcal{O} \subset \mathbb{R}^d$ and $\mathcal{O}' \subset \mathbb{R}^d$, respectively. Furthermore, we assume that $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ are \mathcal{Q} -regular and \mathcal{P} -regular, respectively. Finally, we fix families $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_j)_{j \in J}$ which are L^p -BAPUs simultaneously for all $p \in (0, \infty]$ for \mathcal{Q} and \mathcal{P} , respectively.

Then, we are interested in *sufficient* conditions for existence of an embedding of the form

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z). \quad (1.4)$$

Part of the problem is to formulate an appropriate notion of an embedding for this setting. Necessary conditions are the subject of the next subsection; there, we will see that the sufficient criteria given in this subsection frequently yield *complete characterizations* of the existence of (1.4).

In this introduction, we will only state a few of our results which convey the general flavor. For a more detailed discussion, we refer to the remainder of the paper; in particular to Section 7, in which we summarize the most useful embedding results.

In order to state our results, we need the notion of **almost subordinateness** of two coverings: Recall from above the notation $j^* = \{\ell \in J \mid P_\ell \cap P_j \neq \emptyset\}$ for the set of all (indices of) neighbors of the set P_j . More generally, for $L \subset J$, let $L^* := \bigcup_{\ell \in L} \ell^* \subset J$ and define inductively $L^{0*} := L$ and $L^{(n+1)*} := (L^{n*})^*$. Now, we say that \mathcal{Q} is **almost subordinate to \mathcal{P}** , if there is some $n \in \mathbb{N}_0$ with the following property:

$$\forall i \in I \exists j_i \in J : \quad Q_i \subset P_{j_i}^{n*} \quad \text{where} \quad P_j^{n*} := \bigcup_{\ell \in j^{n*}} P_\ell.$$

The notation Q_i^{n*} (or P_j^{n*}) is one of a few notations that will be used over and over in this paper and is thus worth remembering.

Roughly speaking, the definition of almost subordinateness formalizes the intuition that \mathcal{Q} is “finer than \mathcal{P} ”. Note in particular that we have $\mathcal{O} \subset \mathcal{O}'$ if \mathcal{Q} is almost subordinate to \mathcal{P} .

Finally, we need the notion of **nested sequence spaces**, which we only introduce informally in this introduction. Given a solid sequence space $X \leq \mathbb{C}^K$, an exponent $q \in (0, \infty]$ and for each $k \in K$

some set M_k , as well as a weight $u = (u_{k,m})_{k \in K, m \in M_k}$, we define the space $X([\ell_u^q(M_k)]_{k \in K})$ as the space of all sequences $x = (x_m)_{m \in M}$, with $M := \bigcup_{k \in K} M_k$, for which the (quasi)-norm

$$\|x\|_{X([\ell_u^q(M_k)]_{k \in K})} := \left\| \left(\left\| (u_{k,m} \cdot x_m)_{m \in M_k} \right\|_{\ell^q(M_k)} \right)_{k \in K} \right\|$$

is finite.

Using these notions, we can finally state our first embedding result. It might look daunting at first, but as we will see in Corollary 1.2 below, its assumptions can be greatly simplified in most cases.

Theorem 1.1. Assume that \mathcal{Q} is almost subordinate to \mathcal{P} and define

$$I_j := \{i \in I \mid Q_i \cap P_j \neq \emptyset\} \quad (1.5)$$

for $j \in J$.

Let $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ be \mathcal{Q} -regular and \mathcal{P} -regular, respectively. If $p_1 \leq p_2$ and if the embedding

$$Y \hookrightarrow Z \left(\left[\ell_{|\det T_i|^{p_1^{-1} - p_2^{-1}}}^{p_2'}(I_j) \right]_{j \in J} \right), \quad (1.6)$$

holds, then the map

$$\iota : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z), f \mapsto \sum_{i \in I} \varphi_i f$$

is well-defined and bounded.

Furthermore, $\iota f \in \mathcal{D}'(\mathcal{O}')$ is an extension of $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \leq \mathcal{D}'(\mathcal{O})$. In particular, if $\mathcal{O} = \mathcal{O}'$, then $\iota f = f$ for all $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$. \blacktriangleleft

Note that the main requirement of the preceding theorem is validity of the embedding (1.6). Of course, verifying this condition in general can be difficult. But if $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$ are weighted Lebesgue sequence spaces, we will see in a moment that these conditions can be simplified considerably.

Even further simplifications are possible if w and \mathcal{Q} are **relatively \mathcal{P} -moderate**: For w , this means that the constant

$$C_{w, \mathcal{Q}, \mathcal{P}} := \sup_{j \in J} \sup_{i, \ell \in I_j} \frac{w_i}{w_\ell}$$

is finite. This means that the weight w is “uniformly essentially constant” on the collection of all “small” sets Q_i which intersect the same “large” set P_j . Finally, $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ is called relatively \mathcal{P} -moderate if the weight $(|\det T_i|)_{i \in I}$ is relatively \mathcal{P} -moderate. Roughly speaking, this means that any two “small” sets Q_i, Q_ℓ which intersect the same “large” set P_j have to be of essentially the same measure.

Corollary 1.2. Assume that \mathcal{Q} is almost subordinate to \mathcal{P} and that the weights $w = (w_i)_{i \in I}$ and $v = (v_j)_{j \in J}$ are moderate with respect to \mathcal{Q} and \mathcal{P} , respectively. Finally, assume $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$.

Then, boundedness of the embedding (1.6) is equivalent to finiteness of

$$\left\| \left(v_j \cdot \left\| \left(w_i^{-1} \cdot |\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}} \right)_{i \in I_j} \right\|_{\ell^{p_2' \cdot (q_1/p_2')'}} \right)_{j \in J} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}. \quad (1.7)$$

Furthermore, if w and \mathcal{Q} are relatively \mathcal{P} -moderate and if $\mathcal{O} = \mathcal{O}'$, then finiteness of (1.7) is equivalent to finiteness of

$$\left\| \left(\frac{v_j}{w_{i_j}} \cdot |\det T_{i_j}|^{\frac{1}{p_1} - \left(\frac{1}{p_2} - \frac{1}{q_1} \right)_+ - \frac{1}{p_2}} \cdot |\det S_j|^{\left(\frac{1}{p_2} - \frac{1}{q_1} \right)_+} \right)_{j \in J} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} , \quad (1.8)$$

where for each $j \in J$, an arbitrary $i_j \in I$ with $P_j \cap Q_{i_j} \neq \emptyset$ can be selected. \blacktriangleleft

We remark that the exponent $p \cdot (q/p)'$ uses the convention $p \cdot (q/p)' = \infty$ if $p = \infty$ or if $q \leq p < \infty$. In the remaining cases, we use the usual arithmetic in $(0, \infty]$ to evaluate the product, cf. Lemma 4.8.

Note that the conditions in the preceding theorem, in particular condition (1.8), are usually very easy to check. One simply needs to verify finiteness of the ℓ^q -norm of a *single*, explicitly computable sequence.

Recall that the result from above assumed \mathcal{Q} to be almost subordinate to \mathcal{P} . In the following result, we consider the “reverse” case in which \mathcal{P} is almost subordinate to \mathcal{Q} .

Theorem 1.3. Assume that \mathcal{P} is almost subordinate to \mathcal{Q} and define

$$J_i := \{j \in J \mid P_j \cap Q_i \neq \emptyset\}$$

for $i \in I$.

Let $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ be \mathcal{Q} -regular and \mathcal{P} -regular, respectively. If $p_1 \leq p_2$ and if the embedding

$$Y \left([\ell_u^{p_1^\Delta}(J_i)]_{i \in I} \right) \hookrightarrow Z, \quad (1.9)$$

with

$$u_{i,j} := \begin{cases} |\det S_j|^{\frac{1}{p_2}-1} \cdot |\det T_i|^{1-\frac{1}{p_1}}, & \text{if } p_1 < 1, \\ |\det S_j|^{\frac{1}{p_2}-\frac{1}{p_1}}, & \text{if } p_1 \geq 1 \end{cases}$$

is well-defined and bounded, then we have $\mathcal{O}' \subset \mathcal{O}$ and the map

$$\iota : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z), f \mapsto f|_{C_c^\infty(\mathcal{O}')}$$

is well-defined and bounded. ◀

As above, one can considerably simplify the assumptions of the preceding theorem if Y, Z are both weighted Lebesgue sequence spaces:

Corollary 1.4. Assume that \mathcal{P} is almost subordinate to \mathcal{Q} and that the weights $w = (w_i)_{i \in I}$ and $v = (v_j)_{j \in J}$ are moderate with respect to \mathcal{Q} and \mathcal{P} , respectively. Finally, assume $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$.

Then, condition (1.9) is equivalent to finiteness of

$$\begin{cases} \left\| \left(w_i^{-1} \cdot \left\| \left(|\det S_j|^{\frac{1}{p_1}-\frac{1}{p_2}} \cdot v_j \right)_{j \in J_i} \right\|_{\ell^{q_2 \cdot (p_1^\Delta/q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} & \text{if } p_1 \geq 1, \\ \left\| \left(|\det T_i|^{\frac{1}{p_1}-1} \cdot w_i^{-1} \cdot \left\| \left(|\det S_j|^{1-\frac{1}{p_2}} \cdot v_j \right)_{j \in J_i} \right\|_{\ell^{q_2 \cdot (p_1^\Delta/q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} & \text{if } p_1 < 1. \end{cases} \quad (1.10)$$

Furthermore, if v and \mathcal{P} are relatively \mathcal{Q} -moderate, and if $\mathcal{O} = \mathcal{O}'$, then finiteness of (1.10) is equivalent to finiteness of

$$\left\| \left(\frac{v_{j_i}}{w_i} \cdot |\det S_{j_i}|^{\frac{1}{p_1}-\left(\frac{1}{q_2}-\frac{1}{p_1^\Delta \Delta}\right)_+ - \frac{1}{p_2}} \cdot |\det T_i|^{\left(\frac{1}{q_2}-\frac{1}{p_1^\Delta \Delta}\right)_+} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} , \quad (1.11)$$

where for each $i \in I$, an arbitrary $j_i \in J$ with $Q_i \cap P_{j_i} \neq \emptyset$ can be selected. ◀

We emphasize once more that the preceding two theorems are merely the most accessible of our embedding results. More involved—but more flexible—results are also derived. In particular, we will obtain a variant of the preceding theorems were neither \mathcal{Q} needs to be almost subordinate to \mathcal{P} , nor vice versa. Instead, it suffices if one can write $\mathcal{O} \cap \mathcal{O}' = A \cup B$, where \mathcal{Q} is almost subordinate to \mathcal{P} “near A ” and \mathcal{P} is almost subordinate to \mathcal{Q} “near B ”. Since the precise statement is slightly involved, we don’t spell it out in this introduction. Instead, we refer the interested reader to Theorem 7.5 below.

Finally, there are also results which do not need any subordinateness at all (cf. Theorem 5.6). The preceding theorems, however, have the advantage that they are (relatively) easy to apply and that they are fairly sharp—as we will see now.

1.3.3. Necessary conditions for embeddings. In the previous subsection, we presented various *sufficient* conditions for the existence of an embedding between two decomposition spaces. In the present generality, to my knowledge, these are the best known results—simply because no other results are known which apply in this general setting. Nevertheless, it is crucial to know how *sharp* these sufficient conditions are.

As we will see in this subsection, the results of Corollaries 1.2 and 1.4 are *indeed reasonably sharp*: We will see for $p_2 \in (0, 2] \cup \{\infty\}$ that the (sufficient) conditions given in Theorem 1.1 (and in Corollary 1.2) are also *necessary* conditions for existence of the stated embedding. Likewise, for $p_1 \in [2, \infty]$, the (sufficient) conditions given in Theorem 1.3 (and in Corollary 1.4) are necessary for existence of the embedding. Thus, for these ranges of p_2 or p_1 , we achieve a *complete characterization* of the existence of the embeddings.

Finally, in the case of weighted Lebesgue sequence spaces $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$, this characterization extends to *all* values of p_2 , as long as \mathcal{Q} and w are relatively \mathcal{P} -moderate, or to all values of p_1 , if \mathcal{P} and v are relatively \mathcal{Q} -moderate.

Let us now turn to precise statements. To ensure flexibility, we assume in the following that we are given a set $K \subset \mathcal{O} \cap \mathcal{O}'$, such that the map

$$\iota : \left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)} \right) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z), f \mapsto f \quad (1.12)$$

is well-defined and bounded, where

$$\mathcal{D}_K := \{f \in C^\infty(\mathbb{R}^d) \mid \text{supp } f \subset K\}.$$

In most practical applications, one will choose $K = \mathcal{O} \cap \mathcal{O}'$.

Based on boundedness of ι , we will derive various conditions which are thus necessary conditions for existence of the embedding. For the first such condition, note that Theorems 1.1 and 1.3 always assumed $p_1 \leq p_2$. This assumption is unavoidable:

Theorem 1.5. If the map ι from equation (1.12) is well-defined and bounded and if there are $i \in I$ and $j \in J$ satisfying $K^\circ \cap Q_i \cap P_j \neq \emptyset$ and $\delta_i \in Y$, then $p_1 \leq p_2$.

Furthermore, we necessarily have $\delta_j \in Z$. In case of $p_1 = p_2$, we even have

$$\|\delta_j\|_Z \lesssim \|\delta_i\|_Y, \quad (1.13)$$

where the implied constant is *independent* of i, j . \blacktriangleleft

Remark. In case of $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$, estimate (1.13) simply means $v_j \lesssim w_i$ in case of $K^\circ \cap Q_i \cap P_j \neq \emptyset$. \blacklozenge

For completely general (almost structured) coverings \mathcal{Q}, \mathcal{P} , it is hard to say more. Thus, for our next result, we will assume that \mathcal{Q} —or at least a subfamily $\mathcal{Q}_{I_0} = (Q_i)_{i \in I_0}$ with $I_0 \subset I$ —is almost subordinate to \mathcal{P} , i.e. that for there is a fixed $n \in \mathbb{N}_0$ and for each $i \in I_0$ some $j_i \in J$ satisfying $Q_i \subset P_{j_i}^{n*}$. In this case, much more is true:

Theorem 1.6. Let $\emptyset \neq I_0 \subset I$ and assume that ι as in equation (1.12) is bounded, with $K := \bigcup_{i \in I_0} Q_i$. Furthermore, assume that \mathcal{Q}_{I_0} is almost subordinate to \mathcal{P} .

If there is some $i \in I_0$ with $\delta_i \in Y$, then we have $p_1 \leq p_2$ and the embedding

$$\ell_0(I_0) \cap Y \hookrightarrow Z \left(\left[\ell^{p_2}_{|\det T_i|^{p_1^{-1} - p_2^{-1}}}(I_j \cap I_0) \right]_{j \in J} \right) \quad (1.14)$$

is well-defined and bounded. Here, $\ell_0(I_0) = \langle \delta_i \mid i \in I_0 \rangle \leq \mathbb{C}^I$ is the space of all sequences on I with a finite support which is contained in I_0 . \blacktriangleleft

Remark. For $I_0 = I$, this embedding coincides with the (sufficient) condition given in Theorem 1.1, with the exception that the “inner” norm in Theorem 1.1 is $\ell^{p_2^\vee}$, whereas here, it is ℓ^{p_2} (and we have to restrict to $\ell_0(I_0) = \ell_0(I)$, which is immaterial in most cases).

But for $p_2 \in (0, 2]$, we have $p_2 = p_2^\vee$, so that we get a *complete characterization* of existence of the embedding $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$, at least if \mathcal{Q} is almost subordinate to \mathcal{P} . Furthermore, for $p_2 = \infty$, we will see (cf. Theorem 6.15) that condition (1.14) is satisfied with the “inner norm” $\ell^1 = \ell^{p_2^\vee}$ instead of ℓ^{p_2} , so that the complete characterization holds for all $p_2 \in (0, 2] \cup \{\infty\}$.

Finally, in case of $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$, existence of the embedding (1.14) can again be simplified, similar to condition (1.7). One simply has to replace p_2^∇ by p_2 and I_j by $I_j \cap I_0$ everywhere. \blacklozenge

Of course, the difference between p_2^∇ (in condition (1.6)) and p_2 (in condition (1.14)) is somewhat unsatisfactory. In general, I doubt that it can be removed. Under suitable hypothesis, however, it can:

Theorem 1.7. Assume that \mathcal{Q} is almost subordinate to \mathcal{P} and that ι as in equation (1.12) is bounded, with $K = \mathcal{O} = \mathcal{O}'$. Let $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$, with w, v moderate with respect to \mathcal{Q} and \mathcal{P} , respectively. Finally, assume that \mathcal{Q} and w are relatively \mathcal{P} -moderate.

Then, the quantity in equation (1.8) is finite. \blacktriangleleft

Remark. In view of Corollary 1.2, we have thus achieved a *complete characterization* of existence of the embedding $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$, assuming that \mathcal{Q} is almost subordinate to \mathcal{P} and that \mathcal{Q} and w are relatively \mathcal{P} -moderate. A variant of the above theorem remains true if only a subfamily \mathcal{Q}_{I_0} of \mathcal{Q} is almost subordinate to \mathcal{P} . But for the sake of succinctness, we omit this more general theorem in this introduction. \blacklozenge

Above, we assumed \mathcal{Q} to be almost subordinate to \mathcal{P} . As in the previous subsection, we now consider the “reverse” case in which \mathcal{P} —or at least some subfamily \mathcal{P}_{J_0} of \mathcal{P} —is almost subordinate to \mathcal{Q} . As above, one can obtain a satisfying necessary condition (which is very similar to the sufficient condition from Theorem 1.3) under this assumption:

Theorem 1.8. Let $\emptyset \neq J_0 \subset J$ and assume that the map ι from equation (1.12) is bounded, with $K := \bigcup_{j \in J_0} P_j$. Furthermore, assume that \mathcal{P}_{J_0} is almost subordinate to \mathcal{Q} .

If there is some $i \in I$ with $J_i \cap J_0 \neq \emptyset$, then $p_1 \leq p_2$ and the embedding

$$\eta : \ell_0(J_0) \cap Y \left(\left[\ell_{|\det S_j|^{p_2^{-1}-p_1^{-1}}}^{p_1}(J_i \cap J_0) \right]_{i \in I} \right) \hookrightarrow Z, (x_j)_{j \in J_0} \mapsto (x_j)_{j \in J} \text{ with } x_j = 0 \text{ for } j \in J \setminus J_0 \quad (1.15)$$

is well-defined and bounded. \blacktriangleleft

Remark. For $p_1 \in [2, \infty]$, we have $p_1^\Delta = p_1$ and the weight u defined in Theorem 1.3 satisfies $u_{i,j} = |\det S_j|^{p_2^{-1}-p_1^{-1}}$. Therefore (for $J_0 = J$), the condition given above coincides with the sufficient condition (1.9) from Theorem 1.3 (up to the intersection with $\ell_0(J_0)$), so that we achieve a *complete characterization* of the embedding $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$, as long as $p_1 \in [2, \infty]$ and as long as \mathcal{P} is almost subordinate to \mathcal{Q} . \blacklozenge

Finally, one can again achieve a complete characterization for all $p_1 \in (0, \infty]$ if one is willing to assume slightly more:

Theorem 1.9. Assume that \mathcal{P} is almost subordinate to \mathcal{Q} and that ι as in equation (1.12) is bounded, with $K = \mathcal{O} = \mathcal{O}'$. Furthermore, assume that $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$, with w, v moderate with respect to \mathcal{Q} and \mathcal{P} , respectively. Finally, assume that \mathcal{P} and v are relatively \mathcal{Q} -moderate.

Then, the quantity in equation (1.11) is finite. \blacktriangleleft

Remark. In view of Corollary 1.4, we have thus achieved a *complete characterization* of existence of the embedding $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$, assuming that \mathcal{P} is almost subordinate to \mathcal{Q} and that \mathcal{P} and v are relatively \mathcal{Q} -moderate. Again, a variant of this statement remains true if only a subfamily \mathcal{P}_{J_0} of \mathcal{P} is almost subordinate to and relatively moderate with respect to \mathcal{Q} . \blacklozenge

This concludes the overview of the sufficient conditions and necessary conditions for the existence of *embeddings between decomposition spaces* which are derived in this paper.

Using the same techniques, however, we will derive one more stunning result which nicely illustrates the “rigidity” of decomposition spaces: The mapping $(p, q, w, \mathcal{Q}) \mapsto \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, \ell_w^q)$ is essentially injective. To explain this, we first have to introduce a suitable notion of *equivalence* for coverings. We say that two coverings \mathcal{Q}, \mathcal{P} of the same set $\mathcal{O} \subset \mathbb{R}^d$ are **weakly equivalent** if

$$\sup_{i \in I} |\{j \in J \mid P_j \cap Q_i \neq \emptyset\}| < \infty \quad \text{and} \quad \sup_{j \in J} |\{i \in I \mid Q_i \cap P_j \neq \emptyset\}| < \infty.$$

If \mathcal{Q}, \mathcal{P} consist only of open, connected sets, one can show (cf. Lemma 2.13) that weak equivalence of \mathcal{Q}, \mathcal{P} already implies that \mathcal{Q}, \mathcal{P} are **equivalent**, i.e. that \mathcal{Q} is almost subordinate to \mathcal{P} and vice versa.

Now, we state our “rigidity”-result:

Theorem 1.10. Let \mathcal{Q}, \mathcal{P} be two almost structured coverings of the open set $\mathcal{O} \subset \mathbb{R}^d$ and let $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ be \mathcal{Q} -regular and \mathcal{P} -regular, respectively, with $\ell_0(I) \leq Y$ and with $\ell_0(J) \leq Z$. If

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z) \quad \text{for certain } p_1, p_2 \in (0, \infty],$$

then the following hold:

- (1) We have $p_1 = p_2 =: p$.
- (2) In case of $p \neq 2$, the coverings \mathcal{Q} and \mathcal{P} are weakly equivalent.
- (3) Furthermore, if $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$ for certain $q_1, q_2 \in (0, \infty]$ and certain weights w, v , which are \mathcal{Q} -moderate and \mathcal{P} -moderate, respectively, then the following hold:
 - (a) We have $(p_1, q_1) = (p_2, q_2) =: (p, q)$.
 - (b) We have $w_i \asymp v_j$ if $Q_i \cap P_j \neq \emptyset$.
 - (c) If $(p, q) \neq (2, 2)$, then \mathcal{Q} and \mathcal{P} are weakly equivalent. ◀

It is not possible to remove the assumption $(p, q) \neq (2, 2)$ in the last claim, since an easy application of Plancherel’s theorem shows $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^2, \ell_w^2) = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^2, \ell_v^2)$ for any two almost structured coverings \mathcal{Q}, \mathcal{P} , as long as $w_i \asymp v_j$ if $Q_i \cap P_j \neq \emptyset$, cf. Lemma 6.10 for the precise statement. In this sense, the above theorem is best possible.

1.4. Structure of the paper. For a quick overview of the structure of the paper, we refer to the table of contents on page 2. Here, we give a more detailed overview of the structure and content of the paper:

We begin our exposition in Section 2 with a detailed discussion of different types of coverings. In particular, we recall the class of **structured admissible coverings** as introduced by Borup and Nielsen[4]. We generalize this class of coverings to the two classes of **semi-structured admissible coverings** and **almost structured coverings**. This last-named class has the advantage of being comparatively nonrestrictive—in fact, every covering which we will consider is almost structured—while still possessing all properties needed to obtain well-defined decomposition spaces and to admit satisfactory criteria for the existence of embeddings.

Additionally, in Section 2, we provide a brief overview of the relations between two coverings, including **weak subordinateness** and **almost subordinateness** and the relations between these two concepts. Note that these notions have already been introduced by Feichtinger and Gröbner in their seminal work [7, 6] on decomposition spaces. The final—new—notation which we introduce and explore is that of **relative moderateness** of two coverings.

Next, in Section 3, we begin our study of decomposition spaces, by giving a detailed formal definition of these spaces, including a definition of the partitions of unity which are suitable for defining decomposition spaces. Then, we continue by proving in detail that these spaces are well-defined (i.e., independent of the chosen partition of unity) and complete.

Strictly speaking, none of these results are new or particularly interesting. Nevertheless, they seem important enough to include them here, especially since the (slightly different) definition of decomposition spaces given in [4] yields *incomplete* spaces in general. Furthermore, well-definedness of the decomposition space $\mathcal{D}(\mathcal{Q}, L^p, Y)$ in the Quasi-Banach regime $p \in (0, 1)$ is *not* covered by the original paper [7] on decomposition spaces, since $L^p(\mathbb{R}^d)$ is not a Banach-convolution-module over $L^1(\mathbb{R}^d)$. Instead, one has to rely on certain convolution relations for band-limited L^p -functions, as treated in [21]. We recall these convolution relations in detail, since they are crucial not only for the well-definedness of decomposition spaces, but also for establishing our sufficient criteria for embeddings between decomposition spaces.

In our overview of embedding results (see in particular Theorems 1.1 and 1.3), we saw that the existence of embeddings between decomposition spaces can be reduced (to a certain extent) to the existence of embeddings between certain **nested sequence spaces**. Thus, a good understanding of these nested sequence spaces and their embeddings is crucial. We will obtain this understanding in Section 4, where we study these spaces in detail. In particular, the results derived in that section will

allow us to pass from the abstract criteria in Theorems 1.1 and 1.3 to the concrete ones in Corollaries 1.2 and 1.4—a huge improvement when it comes to applicability of our criteria.

This kind of simplification is a recurring pattern in the present paper: In order to simplify proofs, we formulate our sufficient/necessary criteria in terms of embeddings between nested sequence spaces. Usually, it would be very painful to verify these embeddings directly. But using the results of Section 4, one can then—in a second step—obtain user-friendly versions of these criteria. Thus, although this section is somewhat technical, it is crucial for the overall goal of the paper.

In Section 5, we properly begin our investigation of embeddings between decomposition spaces. We start by estimating the L^p -norm of a sum $\sum_{i \in I} f_i$ of functions which have “almost disjoint” frequency support. Using this estimate and a “dual” version of it, we then derive Theorem 5.6, which provides a very general sufficient criterion for existence of embeddings between two decomposition spaces. Nevertheless, since the prerequisites of this theorem are usually hard to verify directly, we derive several simplified versions of this theorem, which essentially correspond to Theorems 1.1 and 1.3 from above. Finally, we prove Corollary 5.14, which applies if we can write $\mathcal{O} \cap \mathcal{O}' = A \cup B$ in such a way that \mathcal{Q} is almost subordinate to \mathcal{P} “near A ” and vice versa “near B ”.

Next, in Section 6, we investigate sharpness of the sufficient conditions developed in Section 5. The basic idea is to “test” the embedding $\iota : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$ using suitably crafted functions. Depending on the precise assumptions (e.g., \mathcal{Q} is almost subordinate to \mathcal{P} or vice versa), the construction varies. Since these constructions—and the proofs that they yield the desired necessary criteria—are slightly involved, we refrain from describing them in this introduction. Instead, we refer the reader to the beginning of Section 6 itself, where the idea is briefly sketched.

The derivation of the necessary criteria itself is divided into a number of subsections: In subsection 6.1, we begin by proving very elementary conditions: We will see that ι from above can only be bounded if $p_1 \leq p_2$ and if $\|\delta_j\|_Z \lesssim \|\delta_i\|_Y$ for all $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \emptyset$. Although these criteria lack the power of those derived later, they have the crucial advantage that they impose essentially no assumptions on (the relation between) the coverings \mathcal{Q}, \mathcal{P} . In particular, no subordinateness is required. Furthermore, the proofs of these results allow us to illustrate the general proof techniques for deriving necessary criteria, without getting bogged down by additional technical difficulties.

Next, in Subsection 6.2, we prove Theorem 1.10, i.e. we show that $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$ can (essentially) only hold if the “ingredients” of the two spaces are equivalent (in a suitable sense). The proof of this result is technically much more demanding than the proofs from the previous subsection; in fact, many of the proof techniques needed in Subsection 6.3 are already introduced here.

Subsection 6.3 can be considered the heart of Section 6: Here, we provide proofs of Theorems 1.6 and 1.8, i.e., we assume that (a subfamily of) \mathcal{Q} is almost subordinate to \mathcal{P} (or vice versa) and we show that the *sufficient* conditions

$$\begin{aligned} p_1 \leq p_2 \quad \text{and} \quad Y \hookrightarrow Z \left(\left[\ell_{|\det T_i|^{p_1^{-1} - p_2^{-1}}}^{p_2^\nabla}(I_j) \right]_{j \in J} \right) \\ \text{or} \quad p_1 \leq p_2 \quad \text{and} \quad Y \left(\left[\ell_u^{p_1^\Delta}(J_i) \right]_{i \in I} \right) \hookrightarrow Z \end{aligned} \tag{1.16}$$

from Theorems 1.1 and 1.3 are—slightly modified—also *necessary* for the existence of the embedding. The main modification needed is that the “special exponents” p_2^∇ or p_1^Δ are replaced by p_2 or p_1 , respectively. As noted in the remarks after Theorems 1.6 and 1.8, this yields a *complete characterization* of the existence of the desired embedding for $p_2 \in (0, 2]$, or for $p_1 \in [2, \infty]$, respectively.

Of course, the difference between the exponents p_2^∇ and p_1^Δ for the sufficient conditions and the exponents p_2 or p_1 for the necessary conditions is unsatisfactory. In general, I doubt that this gap can be closed; but under the assumption $p_1 = p_2$, we will see in Subsection 6.4 that this gap can at least be *narrowed*: Note that we always have $p_2^\nabla \leq 2$ and $p_1^\Delta \geq 2$. In Subsection 6.4, we will show—for $p_1 = p_2$ —that the sufficient conditions (1.16) are necessary for the existence of the embedding if p_2^∇ and p_1^Δ are replaced by 2. Thus, although our necessary criteria are not able to reach the exponents p_2^∇ and p_1^Δ in general, we see that no sufficient criterion similar to condition (1.16) will ever be able to replace p_2^∇ by something larger than 2, or p_1^Δ by something smaller than 2.

Finally, in Subsection 6.5, we make the additional (rather restrictive) assumption that \mathcal{Q} is relatively \mathcal{P} -moderate (or vice versa). Under this assumption, we achieve two important goals simultaneously:

- We close the offending gap between p_2^∇ and p_2 or between p_1^Δ and p_1 , respectively, thereby obtaining a *complete characterization* for all values of p_2, p_1 .
- We are able to simplify our criteria even further, to the point where only finiteness of a suitable Lebesgue sequence space norm of a *single* sequence needs to be checked, cf. Theorem 1.7 and equation (1.8) or Theorem 1.9 and equation (1.11).

In view of the large number of sufficient or necessary criteria for embeddings between decomposition spaces, we summarize our results in Section 7. In contrast to the preceding sections, where we strove for maximal generality, our aim in this section is ease of applicability, even if this makes our results slightly less general.

Readers who are mainly interested in applying our embedding results are thus encouraged to skip directly to Section 7, possibly after familiarizing themselves with the basic concepts of decomposition spaces outlined in Sections 2–3. In addition to providing a summary of our embedding results, Section 7 also contains a rough “user’s guide”, which should help in deciding for the correct theorem to apply and in verifying the prerequisites of the respective theorem.

In addition to this user’s guide, Section 9 indicates how our results can be applied. Here, we *completely characterize* existence of the embeddings $M_{s_1, \alpha_1}^{p_1, q_1}(\mathbb{R}^d) \hookrightarrow M_{s_2, \alpha_2}^{p_2, q_2}(\mathbb{R}^d)$ between different α -modulation spaces, thereby indicating the power and ease of use of our results. Note that these results greatly extend the state of the art: Up to now, the most comprehensive knowledge about embeddings of α -modulation spaces has been obtained by Han and Wang[13]; they only consider the cases $\alpha_1 \neq \alpha_2$, but $(p_1, q_1) = (p_2, q_2)$ or $(p_1, q_1) \neq (p_2, q_2)$, but $\alpha_1 = \alpha_2$, whereas our criteria apply to arbitrary choices of $\alpha_1, \alpha_2, p_1, p_2, q_1, q_2$.

We remark that α -modulation spaces are particularly straightforward, since the associated covering $\mathcal{Q}^{(\alpha_1)}$ is almost subordinate to and relatively moderate with respect to $\mathcal{Q}^{(\alpha_2)}$ if $\alpha_1 \leq \alpha_2$. As an example where the two coverings in question are more “incompatible”, we also consider embeddings between homogeneous and inhomogeneous Besov spaces, thereby extending previous results of Triebel[22].

The role of Section 8 is somewhat special: As we mentioned above—and as we will see in Example 3.22—using the space of tempered distributions as the “reservoir” for defining decomposition spaces can lead to *incomplete spaces*, even in case of $\bigcup_{i \in I} Q_i = \mathcal{O} = \mathbb{R}^d$. Therefore, we used the reservoirs $\mathcal{D}'(\mathcal{O})$ and $Z'(\mathcal{O}) = \mathcal{F}^{-1}(\mathcal{D}'(\mathcal{O}))$ to define the spaces $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ and $\mathcal{D}(\mathcal{Q}, L^p, Y)$, respectively.

But in order to compare our decomposition spaces to classical function spaces like Besov- and modulation spaces, it is preferable to have a criterion which ensures that $\mathcal{D}(\mathcal{Q}, L^p, Y) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$. Hence, in Section 8, we derive *sufficient* criteria for this to be the case.

2. DIFFERENT CLASSES OF COVERINGS AND THEIR RELATIONS

In this section, we recall the definitions of various existing classes of coverings and introduce two new types of coverings, the (tight) **semi-structured admissible coverings** and the **almost structured coverings**. The first of these two classes will be the main class of coverings that we will consider in the remainder of this paper.

Further, for later use, it will be convenient to have a clear terminology for describing the relations between two coverings, e.g. the fact that one is finer than the other. This terminology and certain auxiliary results are also developed in the present section.

2.1. Admissible and (semi/almost)-structured coverings. We begin with the basic concept of **admissibility** of a covering, which goes back to the original inception of (general) decomposition spaces by Feichtinger and Gröbner in [7]. Admissibility is based on the notion of **neighbors** of a set in a covering, which we introduce first.

Definition 2.1. (cf. [7, Definition 2.3])

Let $X \neq \emptyset$ be a set and assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is a family of subsets of X . For a subset $J \subset I$, we define the **(index)-cluster** of J (also called the **set of neighbors** of J) as

$$J^* := \{i \in I \mid \exists j \in J : Q_i \cap Q_j \neq \emptyset\}.$$

Inductively, we set $J^{0*} := J$ and $J^{(n+1)*} := (J^{n*})^*$ for $n \in \mathbb{N}_0$. We also set $i^{k*} := \{i\}^{k*}$ for $i \in I$ and $k \in \mathbb{N}_0$.

Furthermore, for any subset $J \subset I$, we define

$$Q_J := \bigcup_{j \in J} Q_j$$

and we also introduce the shortcuts $Q_i^{k*} := Q_{i^{k*}}$ and $Q_i^* := Q_{i^*}$ for $i \in I$ and $k \in \mathbb{N}_0$.

If $(\varphi_i)_{i \in I}$ is a family of functions with values in some common topological vector space Y , we define

$$\varphi_J := \sum_{i \in J} \varphi_i$$

for every set $J \subset I$ for which the series converges pointwise. For brevity, we also set $\varphi_i^{k*} := \varphi_{i^{k*}}$ and $\varphi_i^* := \varphi_{i^*}$ for $i \in I$ and $k \in \mathbb{N}_0$. \blacktriangleleft

Remark 2.2. (1) If we are considering two different families $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ with (possibly) nonempty intersection $I \cap J \neq \emptyset$, we will use notation similar to $i^{*\mathcal{Q}}$ and $i^{*\mathcal{P}}$ for $i \in I$ or $i \in J$ to indicate the family which is used to form the respective cluster.

(2) By induction on $n \in \mathbb{N}_0$, it is easy to see for $i, j \in I$ that $i \in j^{n*}$ is equivalent to $j \in i^{n*}$ and that this holds if and only if there is a \mathcal{Q} -chain (cf. [7, Definition 2.3]) of length n from j to i . Here, a finite sequence $(i_\ell)_{\ell=0, \dots, n}$ in I is called a \mathcal{Q} -chain of length n from i_0 to i_n if $Q_{i_{\ell-1}} \cap Q_{i_\ell} \neq \emptyset$ holds for all $\ell \in \underline{n}$.

For further properties of admissible coverings and of the cluster sets J^* , see [7, Lemma 2.1]. \blacklozenge

Now, we can define the notion of an admissible covering and of moderate weights.

Definition 2.3. (cf. [7, Definitions 2.1, 3.1 and 3.2] and [4, Definition 6])

Let $X \neq \emptyset$ be a set and assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is a family of subsets of X . We say that \mathcal{Q} is an **admissible covering** of X , if

- (1) \mathcal{Q} is a covering of X , i.e. $X = \bigcup_{i \in I} Q_i$ and
- (2) $Q_i \neq \emptyset$ for all $i \in I$, and
- (3) the constant $N_{\mathcal{Q}} := \sup_{i \in I} |i^*|$ is finite.

Let $u = (u_i)_{i \in I}$ be a sequence of positive numbers $u_i > 0$. We say that u is a \mathcal{Q} -**moderate weight** if the constant

$$C_{u, \mathcal{Q}} := \sup_{i \in I} \sup_{\ell \in i^*} \frac{u_i}{u_\ell}$$

is finite. \blacktriangleleft

Remark. It is important to observe that the admissibility condition above is dependent on the way in which the covering is indexed using the set I . For example, if one set is “repeated infinitely often”, i.e. if $\emptyset \neq Q = Q_i$ holds for all $i \in I_0 \subset I$, where I_0 is infinite, then this implies $i^* \supset I_0$ for all $i \in I_0$, so that \mathcal{Q} is *not* admissible.

Furthermore, it is worth noting that every admissible covering is of *finite height*. This means that the cardinality $|\{i \in I \mid x \in Q_i\}|$ is bounded uniformly with respect to $x \in X$. More precisely, we have $|\{i \in I \mid x \in Q_i\}| \leq N_{\mathcal{Q}}$ for all $x \in X$. Equivalently, the function $\sum_{i \in I} \mathbb{1}_{Q_i}$ is bounded by $N_{\mathcal{Q}}$. \blacklozenge

As we will see in Section 3, admissibility of the covering \mathcal{Q} is the most basic requirement needed to ensure well-definedness of the decomposition spaces $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$. By well-definedness, we mean that the resulting space does not depend on the chosen partition of unity $\Phi = (\varphi_i)_{i \in I}$. Of course, this can only hold under suitable assumptions on Φ . More precisely, we will assume that $\|\mathcal{F}^{-1}\varphi_i\|_{L^1}$ is uniformly bounded, which ensures well-definedness (at least for $p \geq 1$). Thus, one might ask whether such a partition of unity Φ always exists.

In the following lemma, we develop a first set of *necessary* conditions that the covering \mathcal{Q} has to fulfill if such a partition of unity exists. Note though that we just assume existence of a partition of unity consisting of C_c functions, i.e. we do not yet assume finiteness of $\sup_{i \in I} \|\mathcal{F}^{-1}\varphi_i\|_{L^1}$.

Lemma 2.4. *Let $\mathcal{Q} = (Q_i)_{i \in I}$ be an admissible covering of the open subset $\mathcal{O} \subset X$ of a topological space X . We say that \mathcal{Q} **admits a partition of unity** if there is a family $(\varphi_i)_{i \in I}$ of functions $\varphi_i : X \rightarrow \mathbb{C}$ with*

- (1) $\varphi_i \in C_c(\mathcal{O})$,
- (2) $\varphi_i \equiv 0$ outside of Q_i ,
- (3) $\sum_{i \in I} \varphi_i \equiv 1$ on \mathcal{O} .

In this case, the following hold:

- (1) We have

$$\varphi_M \equiv 1 \quad \text{on } Q_i \text{ for each } i \in I \text{ with } M \supset i^*. \quad (2.1)$$

- (2) We have $\overline{Q_i} \subset \mathcal{O}$ for each $i \in I$. Furthermore, $\overline{Q_i}$ is compact.
- (3) The family of topological interiors $\mathcal{Q}^\circ = (Q_i^\circ)_{i \in I}$ still covers \mathcal{O} .
- (4) The family $(\overline{Q_i})_{i \in I}$ is locally finite in \mathcal{O} (but not necessarily in X). In particular, $(\varphi_i)_{i \in I}$ is a locally finite partition of unity in the sense that $(\text{supp } \varphi_i)_{i \in I}$ is locally finite in \mathcal{O} . \blacktriangleleft

Remark. Observe that $\varphi_M = \sum_{i \in M} \varphi_i$ is well-defined even if M is an infinite set, since at every point at most $N_{\mathcal{Q}}$ summands do not vanish. \blacklozenge

Proof. 1. Since we have $\sum_{j \in I} \varphi_j \equiv 1$ on $\mathcal{O} \supset Q_i$, it suffices to show $\varphi_j \equiv 0$ on Q_i for $j \notin M$. But if $\varphi_j(x) \neq 0$ holds for some $x \in Q_i$, we get $x \in Q_j$ (since φ_j vanishes outside of Q_j) and hence $x \in Q_i \cap Q_j \neq \emptyset$, i.e. $j \in i^* \subset M$. This establishes equation (2.1).

2. By equation (2.1), we have $\sum_{j \in i^*} \varphi_j = \varphi_{i^*} \equiv 1$ on Q_i . But $\varphi_{i^*} \in C_c(\mathcal{O})$, since i^* is finite, because \mathcal{Q} is admissible. Thus, $Q_i \subset \text{supp } \varphi_{i^*} \subset \mathcal{O}$, which entails that $\overline{Q_i} \subset \text{supp } \varphi_{i^*} \subset \mathcal{O}$ is compact.

3. By assumption, $\varphi_i^{-1}(\mathbb{C}^*) \subset Q_i$ for all $i \in I$ and thus $\varphi_i^{-1}(\mathbb{C}^*) \subset Q_i^\circ$ by continuity of φ_i . But we have $\sum_{i \in I} \varphi_i \equiv 1$ on \mathcal{O} and hence

$$\mathcal{O} \subset \bigcup_{i \in I} \varphi_i^{-1}(\mathbb{C}^*) \subset \bigcup_{i \in I} Q_i^\circ \subset \bigcup_{i \in I} Q_i = \mathcal{O}.$$

4. Let $x \in \mathcal{O}$ be arbitrary. Then there is some $i \in I$ with $\varphi_i(x) \neq 0$, since $\sum_{i \in I} \varphi_i \equiv 1$ on \mathcal{O} . Hence, $U_x := \varphi_i^{-1}(\mathbb{C}^*) \subset Q_i^\circ \subset \mathcal{O}$ is an open neighborhood of x . Let $I_x := i^*$. For any index $j \in I$ with $U_x \cap \overline{Q_j} \neq \emptyset$, we have $\emptyset \neq U_x \cap \overline{Q_j} \subset Q_i^\circ \cap \overline{Q_j}$. But this entails $Q_i^\circ \cap Q_j \neq \emptyset$, since otherwise Q_j would be contained in the closed(!) set $(Q_i^\circ)^c$. Thus, $\emptyset \neq Q_i^\circ \cap Q_j \subset Q_i \cap Q_j$ and hence $j \in i^* = I_x$.

Since $I_x = i^*$ is finite by admissibility of \mathcal{Q} and since U_x is an open neighborhood of x , we see that $(\overline{Q_j})_{j \in I_x}$ is a locally finite family in \mathcal{O} . Since φ_i vanishes outside of Q_i , we have $\text{supp } \varphi_i \subset \overline{Q_i}$ for all $i \in I$, so that the last claim follows trivially. \square

As a complement to these *necessary* conditions, we would like to have *sufficient* conditions which ensure existence of a suitable partition of unity $(\varphi_i)_{i \in I}$. To this end, we introduce several more restrictive classes of coverings, the **semi-structured coverings**, the **almost structured coverings** and the **structured coverings**. The general idea of these coverings is—essentially—that each set Q_i should be an affine image $Q_i = T_i Q + b_i$ of a *fixed* set $Q \subset \mathbb{R}^d$. This is the main requirement for a structured covering. For many practical coverings, however, this assumption is somewhat strict: For example, the usual dyadic covering consisting of the dyadic annuli $\left(B_{2^{n+1}}(0) \setminus \overline{B_{2^{n-1}}(0)} \right)_{n \in \mathbb{N}}$ and one ball $B_4(0)$ covering the low frequencies is *not* of this form, since $B_4(0)$ is convex, but the “rings” are not. Thus, the notions of almost structured coverings and semi-structured coverings slightly relax the assumption $Q_i = T_i Q + b_i$ for all $i \in I$, by allowing the set Q to vary—in a controlled way—with $i \in I$.

As we will see later (cf. Theorem 3.19), one can always construct a suitable partition $(\varphi_i)_{i \in I}$ subordinate to any almost structured covering.

We remark that the notion of structured admissible coverings was first introduced (for the case $\mathcal{O} = \mathbb{R}^d$) by Borup and Nielsen in [4, Definition 7] and then slightly generalized by Hartmut Führ and myself in [9, Definition 13] to the definition presented here.

Definition 2.5. Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open and let $I \neq \emptyset$ be an index set. We say that a family $\mathcal{Q} = (Q_i)_{i \in I}$ of subsets $Q_i \subset \mathcal{O}$ is a **semi-structured covering** of \mathcal{O} if for each $i \in I$, there are $T_i \in \text{GL}(\mathbb{R}^d)$ and $b_i \in \mathbb{R}^d$ and an *open* subset $Q'_i \subset \mathbb{R}^d$ with

$$Q_i = T_i Q'_i + b_i$$

and such that the following properties are fulfilled:

- (1) \mathcal{Q} is an admissible covering of \mathcal{O} .
- (2) The sets $(Q'_i)_{i \in I}$ are uniformly bounded, i.e. the following constant is finite:

$$R_{\mathcal{Q}} := \sup_{i \in I} \sup_{x \in Q'_i} |x|.$$

- (3) For “neighboring” indices $i, \ell \in I$, the transformations $T_i \cdot + b_i$ and $T_j \cdot + b_j$ are “uniformly compatible”, i.e. the following constant is finite:

$$C_{\mathcal{Q}} := \sup_{i \in I} \sup_{\ell \in i^*} \|T_i^{-1} T_{\ell}\|.$$

The semi-structured covering \mathcal{Q} is called **tight** if we additionally have the following:

- (4) There is some $\varepsilon > 0$ and for each $i \in I$ some $c_i \in \mathbb{R}^d$ such that $B_{\varepsilon}(c_i) \subset Q'_i$.

A semi-structured covering \mathcal{Q} as above is called **almost structured** if for each $i \in I$, there is an open set $Q''_i \subset \mathbb{R}^d$ such that the following hold:

- (1) We have $\overline{Q''_i} \subset Q'_i$ for all $i \in I$.
- (2) The family $(T_i Q''_i + b_i)_{i \in I}$ is an admissible covering of \mathcal{O} .
- (3) The sets $\{Q'_i \mid i \in I\}$ and $\{Q''_i \mid i \in I\}$ are finite.

Finally, an almost structured covering \mathcal{Q} as above is called a **structured covering** of \mathcal{O} if the set $\{Q'_i \mid i \in I\}$ only consists of one element. \blacktriangleleft

Remark 2.6. (1) If the families $(T_i)_{i \in I}$, $(b_i)_{i \in I}$ and $(Q'_i)_{i \in I}$ satisfy the conditions from above, we call the collection of these three families a **collection of standardizations** for \mathcal{Q} . If we want to emphasize the precise type of conditions from above which are satisfied, we say that the collection of standardizations is tight/almost structured/structured.

- (2) In the remainder of the paper, the phrase “let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ be a [tight] semi-structured covering of \mathcal{O} ” will always implicitly entail that $(T_i)_{i \in I}$, $(b_i)_{i \in I}$ and $(Q'_i)_{i \in I}$ are a [tight] collection of standardizations for \mathcal{Q} . The same holds for “almost structured” or “structured” instead of “semi-structured”.
- (3) If $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ is a *tight* semi-structured covering, we set

$$\varepsilon_{\mathcal{Q}} := \sup \{ \varepsilon > 0 \mid \forall i \in I \exists c_i \in \mathbb{R}^d : B_{\varepsilon}(c_i) \subset Q'_i \}.$$

As we will see below, this supremum is always attained.

- (4) Strictly speaking, the constants $R_{\mathcal{Q}}$, $C_{\mathcal{Q}}$ and $\varepsilon_{\mathcal{Q}}$ depend on the choice of the collection $(T_i)_{i \in I}$, $(b_i)_{i \in I}$ and $(Q'_i)_{i \in I}$ of standardizations for \mathcal{Q} . As above, we will usually suppress this dependence.
- (5) Every almost structured covering is tight, since the family $\{Q'_i \mid i \in I\}$ of open sets is finite for such a covering $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$.
- (6) For brevity, we will use the following convention: If a theorem states that for a given [tight] semi-structured covering² \mathcal{Q} , there is some $C = C(\mathcal{Q}, \dots) > 0$ [or $C = C(\mathcal{Q}, \varepsilon_{\mathcal{Q}}, \dots)$] with a certain property (depending on \mathcal{Q}), then the following is meant: Given $N, R, C > 0$ [and $\varepsilon > 0$], there is a constant $C_0 = C_0(N, R, C, \dots) > 0$ [or $C_0 = C_0(N, R, C, \varepsilon, \dots) > 0$] such that this particular property holds—with C_0 instead of C —for every [tight] semi-structured covering \mathcal{Q} with $N_{\mathcal{Q}} \leq N$, $R_{\mathcal{Q}} \leq R$, $C_{\mathcal{Q}} \leq C$ [and $\varepsilon_{\mathcal{Q}} \geq \varepsilon$].

Note that this means that the constant C may *not* depend on other properties of the covering \mathcal{Q} than on the quantities $N_{\mathcal{Q}}$, $R_{\mathcal{Q}}$, $C_{\mathcal{Q}}$ [and $\varepsilon_{\mathcal{Q}}$], except possibly for those appearing in the “...” part of $C(\mathcal{Q}, \dots)$.

As an example, note that we have $Q_i \subset T_i(B_{R_{\mathcal{Q}}}(0)) + b_i$ for all $i \in I$. Furthermore, for $\ell \in i^*$, Hadamard’s inequality yields

$$|\det T_{\ell}| = |\det T_i| \cdot |\det(T_i^{-1} T_{\ell})| \leq |\det T_i| \cdot \|T_i^{-1} T_{\ell}\|^d \leq C_{\mathcal{Q}}^d \cdot |\det T_i|$$

²Or (almost) structured covering

and hence

$$\begin{aligned}\lambda(Q_\ell) &\leq |\det T_\ell| \cdot \lambda(B_{R_Q}(0)) \\ &\leq \lambda(B_1(0)) \cdot R_Q^d \cdot |\det T_\ell| \\ &\leq \lambda(B_1(0)) \cdot (C_Q R_Q)^d \cdot |\det T_i|.\end{aligned}$$

Using the estimate $|i^*| \leq N_Q$, we finally get

$$\lambda(Q_{i^*}) \leq \sum_{\ell \in i^*} \lambda(Q_\ell) \leq \lambda(B_1(0)) \cdot N_Q \cdot (C_Q R_Q)^d \cdot |\det T_i|.$$

Using the above convention, we have thus shown that there is a constant $C = C(Q, d) > 0$ such that

$$\lambda(Q_{i^*}) \leq C \cdot |\det T_i|$$

holds for all $i \in I$, for every semi-structured covering \mathcal{Q} .

- (7) Finally, we prove that the supremum in the definition of ε_Q is attained. To this end, note that finiteness of R_Q implies that each set Q'_i is bounded, so that $\overline{Q'_i} \subset \mathbb{R}^d$ is compact.

Now, for each $n \in \mathbb{N}$, $i \in I$ and $\varepsilon_n := (1 - \frac{1}{2n})\varepsilon_Q$, there is some $c_n \in \mathbb{R}^d$ satisfying $c_n \in B_{\varepsilon_n}(c_n) \subset Q'_i \subset \overline{Q'_i}$. For some subsequence, we get $c_{n_k} \rightarrow c \in \overline{Q'_i}$ by compactness of $\overline{Q'_i}$.

Finally, let $x \in B_{\varepsilon_Q}(c)$ and set $\delta := \frac{1}{2}(\varepsilon_Q - |x - c|) > 0$. We have

$$|x - c_{n_k}| \xrightarrow{k \rightarrow \infty} |x - c| < \delta + |x - c| < \varepsilon_Q$$

and thus $|x - c_{n_k}| < \delta + |x - c| < \varepsilon_{n_k}$ for sufficiently large $k \in \mathbb{N}$. Hence, $x \in B_{\varepsilon_{n_k}}(c_{n_k}) \subset Q'_i$. We have thus shown $B_{\varepsilon_Q}(c) \subset Q'_i$. \blacklozenge

As our next technical result, we show that the “normalization” $T_i^{-1}(Q_i - b_i) \subset B_R(0)$ —which by definition holds for $R = R_Q$ —remains essentially valid if the set Q_i is replaced by a neighboring set Q_j , as long as one is willing to enlarge the ball $B_R(0)$. This inclusion will become important for establishing well-definedness of $\mathcal{D}_{\mathcal{F}}(Q, L^p, Y)$ in the Quasi-Banach regime $p \in (0, 1)$. In this case, we will see (cf. Theorem 3.4) that the usual estimate

$$\|\mathcal{F}^{-1}(fg)\|_{L^p} \leq \|\mathcal{F}^{-1}f\|_{L^1} \cdot \|\mathcal{F}^{-1}g\|_{L^p}$$

needs to be replaced by

$$\|\mathcal{F}^{-1}(fg)\|_{L^p} \leq [\lambda(K - L)]^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}f\|_{L^p} \cdot \|\mathcal{F}^{-1}g\|_{L^p},$$

where $\text{supp } f \subset K$ and $\text{supp } g \subset L$. Thus, it will be a convenient consequence of the following lemma that we can estimate the Lebesgue measure $\lambda(\overline{Q_i} - \overline{Q_j})$ by a constant multiple of $|\det T_i|$, cf. Corollary 2.8. We remark that the issues related to the constant $[\lambda(\overline{Q_i} - \overline{Q_j})]^{\frac{1}{p}-1}$ in contrast to $|\det T_i|^{\frac{1}{p}-1}$ are somewhat neglected in the treatment of Borup and Nielsen[4].

Lemma 2.7. *Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open and let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ be a semi-structured covering of \mathcal{O} .*

Then the inclusion

$$Q_j \subset T_j(\overline{B_R}(0)) + b_j \subset T_i \left[\overline{B_{(2C_Q+1)R}}(0) \right] + b_i$$

is valid for all $i \in I$, $\ell \in \mathbb{N}_0$ and $j \in i^$, as long as $R > 0$ is chosen such that $Q'_i \subset \overline{B_R}(0)$ holds for all $i \in I$.* \blacktriangleleft

Proof. Let $K := C_Q$. We first prove

$$T_j(\overline{B_R}(0)) + b_j \subset T_i(\overline{B_{(2K+1)R}}(0)) + b_i \quad \forall i \in I \text{ and } j \in i^* \quad (2.2)$$

as long as $\bigcup_{i \in I} Q'_i \subset \overline{B_R}(0)$ holds.

To see this, note that $j \in i^*$ implies $\|T_i^{-1}T_j\| \leq K$, as well as

$$\emptyset \neq Q_i \cap Q_j \subset [T_i(\overline{B_R}(0)) + b_i] \cap [T_j(\overline{B_R}(0)) + b_j],$$

which yields $c_1, c_2 \in \overline{B_R}(0)$ satisfying $T_i c_1 + b_i = T_j c_2 + b_j$. Rearranging results in

$$b_j - b_i = T_i c_1 - T_j c_2 = T_i(c_1 - T_i^{-1}T_j c_2),$$

with

$$|c_1 - T_i^{-1}T_j c_2| \leq |c_1| + \|T_i^{-1}T_j\| |c_2| \leq R + KR = (K+1)R,$$

which implies $T_i^{-1}(b_j - b_i) = c_1 - T_i^{-1}T_j c_2 \in \overline{B_{(K+1)R}}(0)$.

Now, let $b \in \overline{B_R}(0)$ be arbitrary and set $x := T_i^{-1}(T_j b + (b_j - b_i))$. On the one hand,

$$\begin{aligned} |x| &\leq |T_i^{-1}T_j b| + |T_i^{-1}(b_j - b_i)| \\ &\leq \|T_i^{-1}T_j\| |b| + (K+1)R \leq (2K+1)R. \end{aligned}$$

On the other hand, we have $T_i x + b_i = T_j b + (b_j - b_i) + b_i = T_j b + b_j$ and thus

$$T_j b + b_j = T_i x + b_i \in T_i(\overline{B_{(2K+1)R}}(0)) + b_i,$$

which proves the claimed inclusion (2.2).

We can now establish the general claim by induction on $\ell \in \mathbb{N}_0$. The base case $\ell = 0$ is a direct consequence of $Q_i = T_i Q'_i + b_i \subset T_i(\overline{B_R}(0)) + b_i$, since $i^{0*} = \{i\}$.

For the induction step, we note that $j \in i^{(\ell+1)*}$ yields some $k \in i^{\ell*}$ with $j \in k^*$. Let us set $R' := (2K+1)R \geq R$, so that

$$\bigcup_{i \in I} Q'_i \subset \overline{B_R}(0) \subset \overline{B_{R'}}(0)$$

holds. If we apply the induction hypothesis for R' instead of R , we derive

$$T_k(\overline{B_{R'}}(0)) + b_k \subset T_i(\overline{B_{(2K+1)^\ell R'}}(0)) + b_i = T_i(\overline{B_{(2K+1)^{\ell+1}R}}(0)) + b_i.$$

But since $j \in k^*$, we can apply equation (2.2), which results in

$$\begin{aligned} Q_j &\subset T_j(\overline{B_R}(0)) + b_j \\ &\subset T_k(\overline{B_{(2K+1)R}}(0)) + b_k \\ &= T_k(\overline{B_{R'}}(0)) + b_k \\ &\subset T_i(\overline{B_{(2K+1)^{\ell+1}R}}(0)) + b_i. \end{aligned}$$

This completes the induction step. \square

As a corollary, we obtain (a generalization of) the estimate for the Lebesgue measure of the difference set $\overline{Q_i} - \overline{Q_j}$ that was announced above.

Corollary 2.8. *Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open and let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ be a semi-structured covering of \mathcal{O} .*

(1) *There is a constant $C_1 = C_1(d, R_{\mathcal{Q}}) > 0$, such that*

$$\lambda(Q_i) \leq \lambda(\overline{Q_i}) \leq C_1 \cdot |\det T_i|$$

holds for all $i \in I$.

(2) *Conversely, if \mathcal{Q} is tight, there is a constant $C_2 = C_2(d, \varepsilon_{\mathcal{Q}}) > 0$ satisfying*

$$C_2 \cdot |\det T_i| \leq \lambda(Q_i) \leq \lambda(\overline{Q_i})$$

for all $i \in I$.

(3) *Finally, for arbitrary $n \in \mathbb{N}$, there is a constant $C_3 = C_3(\mathcal{Q}, n, d) > 0$, such that*

$$\max_{j \in i^{n*}} \lambda(\overline{Q_i^{n*}} - \overline{Q_j^{n*}}) \leq C_3 \cdot |\det T_i|$$

holds for all $i \in I$. \blacktriangleleft

Remark. For a tight semi-structured covering, we thus get $\lambda(Q_i) \asymp \lambda(\overline{Q_i}) \asymp |\det T_i|$ uniformly with respect to $i \in I$. \blacklozenge

Proof. Let $R := R_{\mathcal{Q}}$. We first observe

$$Q_i \subset \overline{Q_i} = T_i \overline{Q'_i} + b_i \subset T_i(\overline{B_R}(0)) + b_i$$

and hence

$$\lambda(Q_i) \leq \lambda(\overline{Q_i}) \leq |\det T_i| \cdot \lambda(\overline{B_R}(0)),$$

so that we can set $C_1 := \lambda(\overline{B_R(0)}) = v_d \cdot R^d$, where $v_d := \lambda(B_1(0))$ denotes the measure of the d -dimensional Euclidean unit ball.

Conversely, if \mathcal{Q} is tight, set $\varepsilon := \varepsilon_{\mathcal{Q}}$. By part (7) of Remark 2.6, there is for each $i \in I$ some $c_i \in \mathbb{R}^d$ with $B_\varepsilon(c_i) \subset Q'_i$. Hence,

$$Q_i = T_i Q'_i + b_i \supset T_i(B_\varepsilon(c_i)) + b_i,$$

which yields

$$\lambda(\overline{Q_i}) \geq \lambda(Q_i) \geq |\det T_i| \cdot \lambda(B_\varepsilon(c_i)) = v_d \varepsilon^d \cdot |\det T_i|,$$

so that we can set $C_2 := v_d \varepsilon_{\mathcal{Q}}^d$.

It remains to establish the estimate regarding the difference set $\overline{Q_i^{n*}} - \overline{Q_j^{n*}}$ for $j \in i^{n*}$. Note that $j^{n*} \subset i^{2n*}$. Hence, $\overline{Q_i^{n*}} \subset \overline{Q_i^{2n*}}$ and $\overline{Q_j^{n*}} \subset \overline{Q_i^{2n*}}$.

Now, let $K := C_{\mathcal{Q}}$ and $R := R_{\mathcal{Q}}$. By Lemma 2.7, this yields for any $\ell \in i^{2n*}$ the inclusion

$$Q_\ell \subset T_i \left[\overline{B_{(2K+1)^{2n}R}}(0) \right] + b_i.$$

Since $\ell \in i^{2n*}$ was arbitrary, we get

$$\overline{Q_i^{2n*}} \subset T_i \left[\overline{B_{(2K+1)^{2n}R}}(0) \right] + b_i$$

and hence

$$\begin{aligned} \overline{Q_i^{n*}} - \overline{Q_j^{n*}} &\subset \overline{Q_i^{2n*}} - \overline{Q_i^{2n*}} \\ &\subset \left(T_i \left[\overline{B_{(2K+1)^{2n}R}}(0) \right] + b_i \right) - \left(T_i \left[\overline{B_{(2K+1)^{2n}R}}(0) \right] + b_i \right) \\ &\subset T_i \left[\overline{B_{(2K+1)^{2n}R}}(0) - \overline{B_{(2K+1)^{2n}R}}(0) \right] \\ &\subset T_i \left[\overline{B_{(2K+1)^{2n}2R}}(0) \right]. \end{aligned}$$

But this implies

$$\lambda(\overline{Q_i^{n*}} - \overline{Q_j^{n*}}) \leq |\det T_i| \cdot \lambda \left(\overline{B_{(2K+1)^{2n}2R}}(0) \right),$$

so that the choice $C_3 = v_d \cdot \left[(2K+1)^{2n} 2R \right]^d$ is possible. \square

As the final result in this subsection, we use the inclusion from Lemma 2.7 to show that the class of semi-structured coverings is closed under forming clusters, i.e. that $\mathcal{Q}^{k*} := (Q_i^{k*})_{i \in I}$ is again a semi-structured covering of \mathcal{O} if \mathcal{Q} is. This property will frequently be useful, e.g. in the proof of Lemma 2.17 below.

Note that it is *not* true in general that \mathcal{Q}^{k*} is (almost) structured if \mathcal{Q} is. This is one of the main reasons for considering semi-structured coverings and not just almost structured coverings.

Lemma 2.9. *Assume that $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ is a [tight] semi-structured covering of the open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$. Then the following are true:*

- (1) *We have $|i^{k*}| \leq N_{\mathcal{Q}}^k$ for all $i \in I$ and $k \in \mathbb{N}_0$. This even holds if \mathcal{Q} is any admissible family.*
- (2) *The family of k -clusters $\mathcal{Q}^{k*} := (Q_i^{k*})_{i \in I}$ is a [tight] semi-structured covering of \mathcal{O} satisfying $N_{\mathcal{Q}^{k*}} \leq N_{\mathcal{Q}}^{2k+1}$.*
- (3) *There are suitable $P'_i \subset \mathbb{R}^d$ for $i \in I$ so that $(T_i)_{i \in I}, (P'_i)_{i \in I}, (b_i)_{i \in I}$ is a [tight] collection of standardizations for $\mathcal{Q}^{k*} = (T_i P'_i + b_i)_{i \in I}$, with*

$$R_{\mathcal{Q}^{k*}} \leq (2C_{\mathcal{Q}} + 1)^k R_{\mathcal{Q}},$$

$$C_{\mathcal{Q}^{k*}} \leq C_{\mathcal{Q}}^{2k+1}$$

and (in the tight case) with $\varepsilon_{\mathcal{Q}^{k}} \geq \varepsilon_{\mathcal{Q}}$.*

- (4) *If $u = (u_i)_{i \in I}$ is a \mathcal{Q} -moderate weight, then*

$$u_\ell \leq C_{u, \mathcal{Q}}^k \cdot u_i \quad \forall k \in \mathbb{N}_0, i \in I \text{ and } \ell \in i^{k*}.$$

In particular, u is also \mathcal{Q}^{k} -moderate with*

$$C_{u, \mathcal{Q}^{k*}} \leq C_{u, \mathcal{Q}}^{2k+1}.$$

◀

Proof. We first show by induction on $k \in \mathbb{N}_0$ the following: We have

$$\begin{aligned} |i^{k*}| &\leq N_{\mathcal{Q}}^k \text{ for all } i \in I, \\ \text{and } \frac{u_i}{u_j} &\leq C_{u,\mathcal{Q}}^k \text{ for } i \in I, j \in i^{k*}, \\ \text{and } \|T_i^{-1}T_j\| &\leq C_{\mathcal{Q}}^k \text{ for } i \in I, j \in i^{k*}, \end{aligned} \tag{2.3}$$

where $u = (u_i)_{i \in I}$ is \mathcal{Q} -moderate.

The case $k = 0$ is trivial, since $i^{0*} = \{i\}$. For the induction step, first note

$$|i^{(k+1)*}| = \left| \bigcup_{j \in i^{k*}} j^* \right| \leq \sum_{j \in i^{k*}} |j^*| \leq N_{\mathcal{Q}} \cdot |i^{k*}| \leq N_{\mathcal{Q}}^{k+1},$$

where we used the induction hypothesis in the last step. For the second and last estimates, let $j \in i^{(k+1)*}$ be arbitrary. Thus, there is some $\ell \in i^{k*}$ with $j \in \ell^*$. By induction hypothesis, this yields

$$\frac{u_i}{u_j} = \frac{u_i}{u_\ell} \cdot \frac{u_\ell}{u_j} \leq C_{u,\mathcal{Q}}^k \cdot C_{u,\mathcal{Q}} = C_{u,\mathcal{Q}}^{k+1}.$$

Similarly,

$$\|T_i^{-1}T_j\| = \|T_i^{-1}T_\ell T_\ell^{-1}T_j\| \leq \|T_i^{-1}T_\ell\| \cdot \|T_\ell^{-1}T_j\| \leq C_{\mathcal{Q}}^k \cdot C_{\mathcal{Q}} = C_{\mathcal{Q}}^{k+1}.$$

This completes the proof of equation (2.3).

Now, let $i \in I$ and $j \in i^{*\mathcal{Q}^{k*}}$, i.e. with $Q_i^{k*} \cap Q_j^{k*} \neq \emptyset$. This implies $Q_m \cap Q_\ell \neq \emptyset$ for suitable $m \in i^{k*}, \ell \in j^{k*}$ and hence

$$j \in \ell^{k*} \subset m^{(k+1)*} \subset i^{(2k+1)*}.$$

Hence, we have shown $i^{*\mathcal{Q}^{k*}} \subset i^{(2k+1)*\mathcal{Q}}$, which yields

$$|i^{*\mathcal{Q}^{k*}}| \leq |i^{(2k+1)*\mathcal{Q}}| \leq N_{\mathcal{Q}}^{2k+1},$$

so that \mathcal{Q}^{k*} is admissible with $N_{\mathcal{Q}^{k*}} \leq N_{\mathcal{Q}}^{2k+1}$.

As shown above, we also have $u_i/u_j \leq C_{u,\mathcal{Q}}^{2k+1}$ for $j \in i^{*\mathcal{Q}^{k*}} \subset i^{(2k+1)*\mathcal{Q}}$. Thus, u is \mathcal{Q}^{k*} -moderate with $C_{u,\mathcal{Q}^{k*}} \leq C_{u,\mathcal{Q}}^{2k+1}$.

Now, set $P'_i := T_i^{-1}(Q_i^{k*} - b_i)$ for each $i \in I$. Thus, $T_i P'_i + b_i = Q_i^{k*}$ and $P'_i \neq \emptyset$ is open. We now show that $\mathcal{Q}^{k*} = (T_i P'_i + b_i)_{i \in I}$ obeys all requirements from Definition 2.5 for a semi-structured covering.

(1) As seen above, \mathcal{Q}^{k*} is admissible. Now note that $Q_i \neq \emptyset$ implies $i \in i^{k*}$ and hence

$$Q_i \subset Q_i^{k*} = \bigcup_{j \in i^{k*}} Q_j \subset \mathcal{O}$$

for all $i \in I$, so that \mathcal{Q}^{k*} covers \mathcal{O} .

(2) Lemma 2.7 yields

$$Q_i^{k*} = \bigcup_{j \in i^{k*}} Q_j \subset T_i \left(\overline{B_{(2C_{\mathcal{Q}}+1)^k R_{\mathcal{Q}}}}(0) \right) + b_i$$

for all $i \in I$ and thus

$$R_{\mathcal{Q}^{k*}} \leq (2C_{\mathcal{Q}} + 1)^k R_{\mathcal{Q}} < \infty.$$

(3) For $i \in I$ and $j \in i^{*\mathcal{Q}^{k*}} \subset i^{(2k+1)*\mathcal{Q}}$, equation (2.3) yields $\|T_i^{-1}T_j\| \leq C_{\mathcal{Q}}^{2k+1}$. But this implies $C_{\mathcal{Q}^{k*}} \leq C_{\mathcal{Q}}^{2k+1} < \infty$.

Finally, if \mathcal{Q} is a tight covering, there is a family $(c_i)_{i \in I} \in (\mathbb{R}^d)^I$ with $B_{\varepsilon_{\mathcal{Q}}}(c_i) \subset Q'_i$ for all $i \in I$. This yields

$$B_{\varepsilon_{\mathcal{Q}}}(c_i) \subset Q'_i = T_i^{-1}(Q_i - b_i) \subset T_i^{-1}(Q_i^{k*} - b_i) = P'_i$$

for all $i \in I$, so that \mathcal{Q}^{k*} is a tight semi-structured covering with $\varepsilon_{\mathcal{Q}^{k*}} \geq \varepsilon_{\mathcal{Q}}$. \square

2.2. Relations between coverings. The main goal of this paper is to develop criteria for the existence of embeddings $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$. For most of these criteria, in particular for necessary conditions, we will need to impose certain restrictions on the (geometric) relation between the two coverings \mathcal{Q}, \mathcal{P} . In this subsection, we introduce a convenient language for describing such relations. Furthermore, we derive a few consequences of these relations.

We begin by summarizing all possible relations that we consider. We remark that the notions of (weak/almost) subordinateness of coverings were already introduced by Feichtinger and Gröbner, cf. [7, Definition 3.3].

Definition 2.10. (cf. [7, Definition 3.3]) Assume that $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ are two families of subsets of \mathbb{R}^d . Then

- (1) For $i \in I$ we define the **\mathcal{P} -index-cluster** around i (or the **set of \mathcal{P} -neighbors** of i) by

$$J_i := \{j \in J \mid P_j \cap Q_i \neq \emptyset\}.$$

The **\mathcal{Q} -index-cluster** around $j \in J$ is defined analogously and denoted by I_j .

- (2) We say that \mathcal{Q} is **weakly subordinate** to \mathcal{P} if the constant

$$N(\mathcal{Q}, \mathcal{P}) := \sup_{i \in I} |J_i|$$

is finite. For equivalent conditions, see [7, Definition 3.3 and Proposition 3.5].

We say that \mathcal{Q} and \mathcal{P} are **weakly equivalent** if \mathcal{Q} is weakly subordinate to \mathcal{P} and \mathcal{P} is weakly subordinate to \mathcal{Q} .

- (3) We say that \mathcal{Q} is **almost subordinate** to \mathcal{P} if there is a constant $k = k(\mathcal{Q}, \mathcal{P}) \in \mathbb{N}_0$ such that each set Q_i is contained in some P_j^{k*} , i.e. if

$$\forall i \in I \exists j_i \in J : \quad Q_i \subset P_{j_i}^{k*}.$$

If we can take $k(\mathcal{Q}, \mathcal{P}) = 0$, we say that \mathcal{Q} is **subordinate** to \mathcal{P} .

We say that \mathcal{Q} and \mathcal{P} are **equivalent** if \mathcal{Q} is almost subordinate to \mathcal{P} and \mathcal{P} is almost subordinate to \mathcal{Q} .

- (4) A weight $u = (u_i)_{i \in I}$ is called **relatively \mathcal{P} -moderate** if there is a constant $C_{u, \mathcal{Q}, \mathcal{P}} > 0$ with

$$u_i \leq C_{u, \mathcal{Q}, \mathcal{P}} \cdot u_\ell$$

for all $j \in J$ and all $i, \ell \in I$ with $Q_i \cap P_j \neq \emptyset \neq Q_\ell \cap P_j$.

- (5) Now, let $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ be a semi-structured covering of some open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$. We say that \mathcal{Q} is **relatively \mathcal{P} -moderate** if there is a constant $C_{\text{mod}}(\mathcal{Q}, \mathcal{P}) > 0$ satisfying

$$\forall j \in J \forall i, \ell \in I_j : \quad |\det(T_i^{-1} T_\ell)| \leq C_{\text{mod}}(\mathcal{Q}, \mathcal{P}).$$

Since the determinant is multiplicative, an equivalent assumption is that the weight $(|\det T_i|)_{i \in I}$ is relatively \mathcal{P} -moderate. \blacktriangleleft

Remark. We remark that the notations J_i and I_j introduced above are strictly speaking ambiguous, at least in the case $I = J$, i.e. if the *same* index set is used for \mathcal{Q} and \mathcal{P} . Nevertheless, the context will always reveal what is meant.

The condition of relative \mathcal{P} -moderateness of a weight u means that if two sets Q_i, Q_ℓ of the covering \mathcal{Q} are “close” to each other *measured with respect to \mathcal{P}* (i.e., they intersect the same P_j), then u_i, u_ℓ are of similar size. Analogously, \mathcal{Q} is moderate with respect to \mathcal{P} if the determinants $|\det T_i|, |\det T_\ell|$ are of comparable size if Q_i, Q_ℓ are close to each other measured with respect to \mathcal{P} .

It seems that relative moderateness is a novel concept. Its significance will become clear in Subsection 6.5, where we employ it to show that our criteria yield a *complete characterization* of the existence of the embedding $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$ as long as \mathcal{Q} and w are relatively \mathcal{P} -moderate (or as long as \mathcal{P} and v are relatively \mathcal{Q} -moderate). A further (related) application is Lemma 2.17 below, where a convenient estimate of $|J_i|$ is developed, subject to suitable moderateness assumptions. \blacklozenge

The notions introduced in the above definition are of course not independent. In the next lemmata, we explore the connections between these concepts.

Lemma 2.11. *Assume that $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ are two families of nonempty subsets of \mathbb{R}^d . Then the following hold:*

(1) *If \mathcal{Q} is almost subordinate to \mathcal{P} and if \mathcal{P} is admissible, then \mathcal{Q} is weakly subordinate to \mathcal{P} with*

$$N(\mathcal{Q}, \mathcal{P}) \leq N_{\mathcal{P}}^{k(\mathcal{Q}, \mathcal{P})+1}.$$

If \mathcal{Q}, \mathcal{P} are coverings of $\mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$, respectively, then we also have $\mathcal{O} \subset \mathcal{O}'$.

(2) *If $Q_i \subset P_j^{k*}$ holds for some $i \in I$, $j \in J$ and $k \in \mathbb{N}_0$, then $J_i \subset \ell^{(2k+2)*}$ holds for all $\ell \in J_i$. In particular, $Q_i \subset P_{\ell}^{(2k+2)*}$ for all $\ell \in J_i$.*

(3) *If \mathcal{Q} is almost subordinate to \mathcal{P} , then $Q_i \subset P_j^{(2k(\mathcal{Q}, \mathcal{P})+2)*}$ holds for all $i \in I$ and all $j \in J_i$. ◀*

Remark. The inclusion $Q_i \subset P_{\ell}^{(2k+2)*}$ for all $\ell \in J_i$ from part (2) of the Lemma will turn out to be very useful for the proofs of our embedding results in Sections 5 and 6: By only assuming $Q_i \subset P_j^{k*}$ for some $j \in J$, we can already conclude $Q_i \subset P_{\ell}^{m*}$ (for suitable $m > k$) for all $\ell \in J$ which intersect Q_i non-trivially. ♦

Proof. 1. Choose $k := k(\mathcal{Q}, \mathcal{P})$ so that for each $i \in I$ there is some $j_i \in J$ with $Q_i \subset P_{j_i}^{k*}$. Now let $j \in J_i$ for some $i \in I$. This implies

$$\emptyset \neq P_j \cap Q_i \subset P_j \cap P_{j_i}^{k*}$$

and hence $j \in j_i^{(k+1)*}$. Thus, $J_i \subset j_i^{(k+1)*}$, so that Lemma 2.9 yields $|J_i| \leq |j_i^{(k+1)*}| \leq N_{\mathcal{P}}^{k+1}$. Since this holds for all $i \in I$, we get

$$N(\mathcal{Q}, \mathcal{P}) = \sup_{i \in I} |J_i| \leq N_{\mathcal{P}}^{k+1} < \infty,$$

as desired.

Finally, assume that \mathcal{Q}, \mathcal{P} are coverings of $\mathcal{O}, \mathcal{O}'$, respectively. Then

$$\mathcal{O} = \bigcup_{i \in I} Q_i \subset \bigcup_{i \in I} P_{j_i}^{k*} \subset \bigcup_{j \in J} P_j \subset \mathcal{O}'.$$

2. Fix $\ell \in J_i$ and let $m \in J_i$ be arbitrary. This implies

$$\emptyset \neq Q_i \cap P_s \subset P_j^{k*} \cap P_s \text{ for } s \in \{m, \ell\}$$

and thus $j \in \ell^{(k+1)*}$, as well as $m \in j^{(k+1)*} \subset \ell^{(2k+2)*}$. We have thus shown $J_i \subset \ell^{(2k+2)*}$. Finally, note that $Q_i \subset P_j^{k*}$ and $J_i \subset \ell^{(2k+2)*}$ imply

$$Q_i \subset \bigcup_{m \in J} (P_m \cap Q_i) \subset \bigcup_{m \in J_i} P_m \subset P_{\ell}^{(2k+2)*}$$

for all $\ell \in J_i$.

3. This is a special case of part (2). ◻

It was observed by Feichtinger and Gröbner in [7, Proposition 3.6] that weak subordinateness implies almost subordinateness if we impose certain *connectivity assumptions*. This will turn out to be very convenient for verifying almost subordinateness for concrete examples.

The statement of the following lemma is very close to that of [7, Proposition 3.6]. The only difference is that a few unnecessary assumptions (like connectedness of the space X) have been removed and the statement of the lemma has been made more quantitative. The proof is still the same as that of [7, Proposition 3.6] and is hence omitted.

Lemma 2.12. (cf. [7, Proposition 3.6]) *Let $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ be families of subsets of a topological space X .*

Let $i \in I$ and assume that Q_i is path-connected with $Q_i \subset \bigcup_{j \in J} P_j$ and so that

$$J_i := \{j \in J \mid P_j \cap Q_i \neq \emptyset\}$$

is finite. Furthermore, assume that P_j is open for every $j \in J_i$.

Let $r := |J_i|$. We then have $J_i \subset j^{r}$ and in particular $Q_i \subset P_j^{r*}$ for every $j \in J_i$. ◀*

The above lemma immediately yields the following corollary.

Corollary 2.13. *Let $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ be families of subsets of \mathbb{R}^d such that each Q_i is path-connected and such that each P_j is open.*

Assume that \mathcal{Q} is weakly subordinate to \mathcal{P} with $\bigcup_{i \in I} Q_i \subset \bigcup_{j \in J} P_j$. Then \mathcal{Q} is almost subordinate to \mathcal{P} with $k(\mathcal{Q}, \mathcal{P}) \leq N(\mathcal{Q}, \mathcal{P})$. ◀

One of the fundamental tools that we will use again and again in the remainder of the paper is the following **disjointization lemma** for admissible coverings that was developed by Feichtinger and Gröbner in [7, Lemma 2.9].

Lemma 2.14. (cf. [7, Lemma 2.9]) *Let $\mathcal{O} \neq \emptyset$ be a set and assume that the family $\mathcal{Q} = (Q_i)_{i \in I}$ is an admissible covering of \mathcal{O} .*

Let $m \in \mathbb{N}_0$ be arbitrary. Then there exists a finite partition $I = \bigsqcup_{r=1}^{r_0} I^{(r)}$ with $Q_i^{m} \cap Q_j^{m*} = \emptyset$ for all $i, j \in I^{(r)}$ with $i \neq j$ and all $r \in \{1, \dots, r_0\}$. In fact, one can choose $r_0 = N_{\mathcal{Q}}^{2m+1}$.* ◀

Note that the exact choice of r_0 from above is not explicitly stated in [7, Lemma 2.9]. For the sake of completeness, we thus present a proof which is based on the following slightly more general lemma. The proof, however, is still similar to that of [7, Lemma 2.9].

Lemma 2.15. *Let $X \neq \emptyset$ be a set and let $\sim \subset X \times X$ be a relation which is reflexive and symmetric, but not necessarily transitive.*

For $x \in X$, let $[x] := \{y \in X \mid y \sim x\}$ and assume that the maximal cardinality $N := \sup_{x \in X} |[x]|$ is finite.

Then there is a partition $X = \bigsqcup_{\ell=1}^N X_\ell$ with $x \not\sim y$ for all $x, y \in X_\ell$ with $x \neq y$ and arbitrary $\ell \in \underline{N}$. ◀

Proof. Let $X_1 \subset X$ be maximal with the following property: For $x, y \in X_1$ with $x \neq y$, we have $x \not\sim y$. Existence of such a set is an easy consequence of Zorn's Lemma.

Now, if X_1, \dots, X_m are already constructed for some $m \in \underline{N-1}$, let $X_{m+1} \subset X \setminus (X_1 \cup \dots \cup X_m)$ be maximal with the same property as above. Again, existence follows from Zorn's Lemma.

Clearly, $(X_\ell)_{\ell \in \underline{N}}$ are pairwise disjoint and satisfy the property that $x \not\sim y$ for $x, y \in X_\ell$ with $x \neq y$ and arbitrary $\ell \in \underline{N}$.

It remains to show $X = \bigcup_{\ell=1}^N X_\ell$. Suppose that this fails. Then there is some $x \in X \setminus \bigcup_{\ell=1}^N X_\ell$. Thus, $X_\ell \cup \{x\} \subset X \setminus (X_1 \cup \dots \cup X_{\ell-1})$ is a strict superset of X_ℓ for arbitrary $\ell \in \underline{N}$. By maximality of X_ℓ , we see that there must be some $y_\ell \in X_\ell \cup \{x\}$ with $x \neq y_\ell$ and $x \sim y_\ell$. Note $y_\ell \in X_\ell$ since $x \neq y_\ell$. By disjointness of the $(X_\ell)_{\ell \in \underline{N}}$, we see that x, y_1, \dots, y_N are pairwise distinct. Hence,

$$N \geq |[x]| \geq |\{x, y_1, \dots, y_N\}| = N + 1,$$

a contradiction. ◻

Proof of Lemma 2.14. For $i, j \in I$, write $i \sim j$ if and only if $Q_i^{m*} \cap Q_j^{m*} \neq \emptyset$. Note that $Q_i^{m*} \supset Q_i \neq \emptyset$ for all $i \in I$, by definition of an admissible covering. This shows that \sim is reflexive. Symmetry of \sim is clear.

Finally, note (in the notation of Lemma 2.15) that $[i] = i^{*\mathcal{Q}^{m*}}$ satisfies

$$|[i]| = |i^{*\mathcal{Q}^{m*}}| \leq N_{\mathcal{Q}^{m*}} \leq N_{\mathcal{Q}}^{2m+1},$$

thanks to Lemma 2.9. Application of Lemma 2.15 completes the proof. ◻

The next lemma shows that restricting attention solely to coverings for which one is relatively moderate with respect to the other would prevent one from handling the case where the two covered sets are distinct.

Lemma 2.16. *Let $\emptyset \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open and assume that $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ is a semi-structured admissible covering of \mathcal{O} and that $\mathcal{P} = (P_j)_{j \in J}$ is an admissible covering of \mathcal{O}' .*

Finally, assume $\mathcal{O}' \cap \partial \mathcal{O} \neq \emptyset$ and that \mathcal{Q}, \mathcal{P} admit partitions of unity, cf. Lemma 2.4. Then the following hold:

- (1) \mathcal{P} is not weakly subordinate to \mathcal{Q} . Even more, we have $|I_j| = \infty$ for some $j \in J$.
- (2) If \mathcal{Q} is tight and almost subordinate to \mathcal{P} , then \mathcal{Q} is not relatively \mathcal{P} -moderate. ◀

Remark. It is instructive to note that $\mathcal{O}' \cap \partial\mathcal{O} \neq \emptyset$ is *always* satisfied if $\mathcal{O} \subset \mathbb{R}^d$ is dense (e.g. of full measure) with $\mathcal{O} \subsetneq \mathcal{O}'$. In particular, this holds if \mathcal{P} is the inhomogeneous Besov covering of $\mathcal{O}' = \mathbb{R}^d$ and if \mathcal{Q} denotes the homogeneous Besov covering of $\mathcal{O} = \mathbb{R}^d \setminus \{0\}$. \blacklozenge

Proof. Choose $x \in \mathcal{O}' \cap \partial\mathcal{O}$. By Lemma 2.4, there is some $j \in J$ with $x \in P_j^\circ$. Because of $x \in \partial\mathcal{O}$, there is a sequence $(x_n)_{n \in \mathbb{N}} \in \mathcal{O}^\mathbb{N}$ with $x_n \xrightarrow{n \rightarrow \infty} x$. Note that we have $x_n \in P_j^\circ$ for n sufficiently large, so that we can assume $x_n \in P_j^\circ$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, choose $i_n \in I$ with $x_n \in Q_{i_n}$. Assume towards a contradiction that there is some $i \in I$ with $i_n = i$ for infinitely many $n \in \mathbb{N}$. Since $\overline{Q_i} \subset \mathcal{O}$ holds by Lemma 2.4, this implies $x \in \overline{Q_i} \subset \mathcal{O} = \mathcal{O}^\circ$ in contradiction to $x \in \partial\mathcal{O}$. Thus, restricting to a subsequence, we can assume that the $(i_n)_{n \in \mathbb{N}}$ are pairwise distinct. Using $x_n \in Q_{i_n} \cap P_j^\circ$, we conclude that $I_j \supset \{i_n \mid n \in \mathbb{N}\}$ is infinite, so that \mathcal{P} is not weakly subordinate to \mathcal{Q} . This establishes the first claim.

Now assume towards a contradiction that \mathcal{Q} is tight, almost subordinate to \mathcal{P} and relatively \mathcal{P} -moderate. Using part (2) of Lemma 2.11 and the definition of relative \mathcal{P} -moderateness, we see that there is some $\ell \in \mathbb{N}$ with $Q_{i_n} \subset P_j^{\ell*}$ for all $n \in \mathbb{N}$ and some $C > 0$ with

$$|\det(T_{i_n}^{-1}T_{i_m})| \leq C$$

for all $n, m \in \mathbb{N}$ because of $Q_{i_n} \cap P_j \neq \emptyset$.

Since \mathcal{Q} is tight, Corollary 2.8 yields a constant $c = c(\mathcal{Q}, \varepsilon_{\mathcal{Q}}, d) > 0$ satisfying

$$\lambda(Q_{i_n}) \geq c \cdot |\det T_{i_n}| \geq \frac{c}{C} \cdot |\det T_{i_1}| =: c'$$

for all $n \in \mathbb{N}$.

But admissible coverings are of finite height; explicitly, we have $\sum_{n \in \mathbb{N}} \chi_{Q_{i_n}} \leq \sum_{i \in I} \chi_{Q_i} \leq N_{\mathcal{Q}}$. But above, we showed $Q_{i_n} \subset P_j^{\ell*}$ for all $n \in \mathbb{N}$. All in all, we conclude

$$\sum_{n \in \mathbb{N}} \chi_{Q_{i_n}} \leq N_{\mathcal{Q}} \cdot \chi_{\overline{P_j^{\ell*}}}.$$

Integration yields

$$\infty > N_{\mathcal{Q}} \cdot \lambda(\overline{P_j^{\ell*}}) \geq \sum_{n \in \mathbb{N}} \lambda(Q_{i_n}) \geq \sum_{n \in \mathbb{N}} c' = \infty,$$

a contradiction. Here, the finite union $\overline{P_j^{\ell*}} = \bigcup_{m \in j^{\ell*}} \overline{P_m}$ is of finite measure, by compactness of each $\overline{P_m}$, cf. Lemma 2.4. \square

Using similar techniques as in the proof above, we will now establish an easy way to calculate the cardinality $|I_j|$ of the set of \mathcal{Q} -neighbors of P_j . This method of estimating $|I_j|$ is implicitly used in [13] for the concrete setting of α -modulation spaces, but not stated explicitly. In the present generality, it seems to be a new observation which will be of great use to us in Subsection 6.5.

It is important to note that we assume \mathcal{Q} to be relatively \mathcal{P} -moderate and (for the lower estimate) almost subordinate to \mathcal{P} . Without these assumptions, it is easy to see that an estimate as in equation (2.5) below fails in general.

Lemma 2.17. *Let $\emptyset \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open and assume that the families $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J} = (S_j P'_j + c_j)_{j \in J}$ are tight semi-structured coverings of \mathcal{O} and \mathcal{O}' , respectively.*

Finally, let $I_0 \subset I$, $J_0 \subset J$ and assume that

- (1) $\mathcal{Q}_{I_0} := (Q_i)_{i \in I_0}$ is almost subordinate to \mathcal{P} .
- (2) \mathcal{Q}_{I_0} is relatively \mathcal{P}_{J_0} -moderate, with $\mathcal{P}_{J_0} := (P_j)_{j \in J_0}$.
- (3) There is some $r \in \mathbb{N}_0$ and some $C_0 > 0$ such that

$$\lambda(P_j) \leq C_0 \cdot \lambda\left(\bigcup_{i \in I_0 \cap I_j} Q_i^{r*}\right) \tag{2.4}$$

holds for all $j \in J_0$ with $1 \leq |I_0 \cap I_j| < \infty$.

Then there are positive constants

$$C_1 = C_1(d, \mathcal{Q}, \varepsilon_{\mathcal{Q}}, \mathcal{P}, k(\mathcal{Q}_{I_0}, \mathcal{P}), C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}_{J_0}))$$

and

$$C_2 = C_2(d, C_0, r, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{P}}, C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}_{J_0}))$$

with

$$C_1^{-1} \cdot |I_0 \cap I_j| \leq \frac{|\det S_j|}{|\det T_i|} \leq C_2 \cdot |I_0 \cap I_j| \quad \forall i \in I_0 \cap I_j \quad (2.5)$$

for all $j \in J_0$. Finally,

(1) Tightness of \mathcal{P} and assumption (3) are only needed for the right estimate in equation (2.5).

(2) Tightness of \mathcal{Q} and assumption (1) are only needed for the left estimate in equation (2.5). ◀

Remark. 1. The moderateness assumption means explicitly that there is a constant $C = C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}_{J_0})$ with $|\det T_i| \leq C \cdot |\det T_\ell|$ for all $i, \ell \in I_0$ for which there is some $j \in J_0$ with $Q_i \cap P_j \neq \emptyset \neq Q_\ell \cap P_j$.

2. It is worth noting that estimate (2.4) from above is trivially satisfied (with $C_0 = 1$, $r = 0$ and arbitrary $J_0 \subset J$) if we have $\mathcal{O}' \subset \mathcal{O}$ (up to a set of measure zero) and $I_0 = I$, because in that case we have (up to a set of measure zero)

$$P_j \subset \mathcal{O}' \subset \mathcal{O} = \bigcup_{i \in I} Q_i = \bigcup_{i \in I_0} Q_i^{0*}$$

which easily implies $P_j \subset \bigcup_{i \in I_0 \cap I_j} Q_i^{0*}$ (up to a set of measure zero) and thus $\lambda(P_j) \leq \lambda\left(\bigcup_{i \in I_0 \cap I_j} Q_i^{0*}\right)$.

3. Finally, it is important to observe that the statement in equation (2.5) is void for $j \in J_0$ with $I_0 \cap I_j = \emptyset$. ♦

Proof. Corollary 2.8 ensures existence of constants $c = c(d, \varepsilon_{\mathcal{Q}}) > 0$ and $C = C(d, R_{\mathcal{Q}}) > 0$ with

$$c^{-1} \cdot |\det T_i| \leq \lambda(Q_i) \leq C \cdot |\det T_i| \quad (2.6)$$

for all $i \in I$. The same corollary also yields $c_1 = c_1(d, \varepsilon_{\mathcal{P}}) > 0$ and $C_1 = C_1(d, R_{\mathcal{P}}) > 0$ so that an analogous estimate holds for \mathcal{P} .

Define $K := C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}_{J_0})$, so that $|\det(T_i^{-1}T_\ell)| \leq K$ holds for all $j \in J_0$ and $i, \ell \in I_j \cap I_0$. Lemma 2.9 shows that \mathcal{Q}^{r*} is also a tight semi-structured covering with $\varepsilon_{\mathcal{Q}^{r*}} \geq \varepsilon_{\mathcal{Q}}$ and $R_{\mathcal{Q}^{r*}} \leq (2C_{\mathcal{Q}} + 1)^r R_{\mathcal{Q}}$, so that equation (2.6) is also valid for \mathcal{Q}^{r*} with (possibly different) constants $c' = c'(d, \varepsilon_{\mathcal{Q}}) > 0$ and $C' = C'(d, r, \mathcal{Q}) > 0$.

Choose an arbitrary $j \in J_0$ with $I_0 \cap I_j \neq \emptyset$ (in case of $I_0 \cap I_j = \emptyset$, the claim is void). Fix any $i_0 \in I_0 \cap I_j$. This yields $|\det T_i| \leq K \cdot |\det T_{i_0}|$ for all $i \in I_j \cap I_0$. If $|I_0 \cap I_j|$ is infinite, the upper estimate in equation (2.5) is trivial. Hence, we can assume that $I_0 \cap I_j$ is finite, i.e. $1 \leq |I_0 \cap I_j| < \infty$. By equation (2.4), this implies

$$\begin{aligned} c_1^{-1} \cdot |\det S_j| &\leq \lambda(P_j) \leq C_0 \cdot \lambda\left(\bigcup_{i \in I_0 \cap I_j} \overline{Q_i^{r*}}\right) \\ &\leq C_0 \cdot C' \cdot \sum_{i \in I_0 \cap I_j} |\det T_i| \\ &\leq C_0 \cdot C' \cdot K \cdot |\det T_{i_0}| \cdot |I_0 \cap I_j|. \end{aligned}$$

Since $i_0 \in I_0 \cap I_j$ was arbitrary, the upper estimate in equation (2.5) is established.

For the lower estimate, note that Lemma 2.11 yields $Q_i \subset P_j^{\ell*}$ for all $i \in I_0$ and $j \in J_i$, where we defined $\ell := 2k(\mathcal{Q}_{I_0}, \mathcal{P}) + 2$. Furthermore, Lemma 2.9 shows that $\mathcal{P}^{\ell*}$ is a tight semi-structured covering, so that equation (2.6) also holds for $\mathcal{P}^{\ell*}$ with constants $c'_1 > 0$ and $C'_1 = C'_1(\mathcal{P}, d, \ell) > 0$.

Finally, admissibility of \mathcal{Q} entails $\sum_{i \in I} \chi_{Q_i} \leq N_{\mathcal{Q}}$ and hence $\sum_{i \in I_0 \cap I_j} \chi_{Q_i}(\xi) \leq N_{\mathcal{Q}} \cdot \chi_{\overline{P_j^{\ell*}}}(\xi)$ for all $j \in J$ and $\xi \in \mathbb{R}^d$. Now, let $j \in J_0$ be arbitrary, let $\Gamma \subset I_0 \cap I_j$ be a finite subset and let $i_0 \in I_0 \cap I_j$

be arbitrary. We have

$$\begin{aligned}
 \frac{1}{cK} \cdot |\det T_{i_0}| \cdot |\Gamma| &\leq \sum_{i \in \Gamma} c^{-1} \cdot |\det T_i| \\
 &\leq \sum_{i \in \Gamma} \lambda(Q_i) \\
 &= \int_{\mathbb{R}^d} \sum_{i \in \Gamma} \chi_{Q_i}(\xi) \, d\xi \\
 &\leq N_{\mathcal{Q}} \cdot \int_{\mathbb{R}^d} \chi_{P_j^{\ell*}}(\xi) \, d\xi \\
 &\leq N_{\mathcal{Q}} C'_1 \cdot |\det S_j|.
 \end{aligned}$$

Since $\Gamma \subset I_0 \cap I_j$ was an arbitrary finite subset, we see that $I_0 \cap I_j$ is finite and that the lower bound in equation (2.5) is satisfied. \square

3. (FOURIER-SIDE) DECOMPOSITION SPACES

Now that we understand the relevant types of coverings, we are in a position to properly start our analysis of decomposition spaces. In this section, we will define these spaces and derive their most important basic properties, i.e. well-definedness and completeness.

We note that these are nontrivial issues: Completeness only holds in general when the right “reservoir” of functions is used. We will see that, in general, the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions is *not* a suitable choice, even for $\mathcal{O} = \mathbb{R}^d$. Furthermore, for $p \in (0, 1)$, independence of $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ from the chosen partition of unity is not as straightforward as for $p \in [1, \infty]$, since Young’s convolution inequality $L^1 * L^p \hookrightarrow L^p$ fails in the Quasi-Banach regime $p \in (0, 1)$. Instead, we have

$$\|\mathcal{F}^{-1}(fg)\|_{L^p} \lesssim \|\mathcal{F}^{-1}f\|_{L^p} \cdot \|\mathcal{F}^{-1}g\|_{L^p} \quad (3.1)$$

if both f, g are compactly supported. Somewhat unexpectedly, the implied constant in (3.1) depends on the measure of the *algebraic difference* (or *Minkowski difference*)

$$K - L = \{k - \ell \mid k \in K, \ell \in L\},$$

where $\text{supp } f \subset K$ and $\text{supp } g \subset L$. Both of these issues are somewhat neglected in the standard reference [4] for Fourier-analytic decomposition spaces.

The structure of this section is as follows: In the first subsection, we briefly present a treatment of a variant of Young’s inequality for convolution in $L^p(\mathbb{R}^d)$ in the Quasi-Banach regime $p \in (0, 1)$. This is based on Triebel’s book [21]. We repeat some of the arguments given there, since we need to know the explicit constant arising in equation (3.1), depending on the supports of f, g . This constant is not stated explicitly by Triebel.

The remaining subsections 3.2–3.4 are devoted, respectively, to the definition, well-definedness and completeness of the (Fourier side) decomposition space $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ and its “space-side” counterpart. While defining these spaces, we also introduce the notion of L^p -**BAPUs**, i.e. of the class of partitions of unity $(\varphi_i)_{i \in I}$ which are suitable to define these spaces. In particular, we note that every almost-structured covering admits an L^p -BAPU.

3.1. Convolution relations for L^p , $p \in (0, 1)$. As noted above, in this subsection, we will establish the inequality

$$\|\mathcal{F}^{-1}(f \cdot g)\|_{L^p} \leq C \cdot \|\mathcal{F}^{-1}f\|_{L^p} \cdot \|\mathcal{F}^{-1}g\|_{L^p} \quad (3.2)$$

for compactly supported functions/distributions f, g and $p \in (0, 1)$. Our treatment is largely identical to that of Triebel in [21], but we still include the proofs, since the dependency of the constant C on the supports of f, g will be relevant to us. This dependency is not explicitly stated by Triebel.

We first note that Young’s inequality $\|f * g\|_{L^p} \leq \|f\|_{L^1} \cdot \|g\|_{L^p}$ fails completely for $p \in (0, 1)$, even if both f, g have compact Fourier support. This is shown in the following example:

Example 3.1. Let us define

$$g_1 : \mathbb{R} \rightarrow \mathbb{R}, \xi \mapsto \max \{0, 1 - |\xi|\} = \begin{cases} 0, & \text{if } \xi \leq -1, \\ 1 + \xi, & \text{if } -1 \leq \xi \leq 0, \\ 1 - \xi, & \text{if } 0 \leq \xi \leq 1, \\ 0, & \text{if } \xi \geq 1 \end{cases}$$

and

$$g_2 : \mathbb{R} \rightarrow \mathbb{R}, \xi \mapsto \begin{cases} 0, & \text{if } \xi \leq -1, \\ (\xi - 1)^2 \cdot (\xi + 1)^2, & \text{if } -1 \leq \xi \leq 1, \\ 0, & \text{if } \xi \geq 1. \end{cases}$$

Then g_1, g_2 are supported in $[-1, 1]$. Hence, if we set $f_j := \mathcal{F}^{-1}g_j$ for $j \in \{1, 2\}$, then f_1, f_2 are bandlimited.

A straightforward, but tedious calculation shows

$$f_1(x) = -\frac{1}{2} \frac{\cos(2\pi x) - 1}{\pi^2 x^2},$$

as well as

$$f_2(x) = \frac{3}{2} \frac{\sin(2\pi x)}{\pi^5 x^5} - 3 \frac{\cos(2\pi x)}{\pi^4 x^4} - 2 \frac{\sin(2\pi x)}{\pi^3 x^3}$$

for $x \in \mathbb{R} \setminus \{0\}$.

Being (inverse) Fourier transforms of L^1 -functions, f_1, f_2 are both bounded. Furthermore, f_1 decays (at least) like x^{-2} and f_2 decays (at least) like $|x|^{-3}$ at $\pm\infty$. This implies $f_1 \in L^p(\mathbb{R})$ for all $p > \frac{1}{2}$ as well as $f_2 \in L^p(\mathbb{R})$ for $p > \frac{1}{3}$. In particular, $f_1, f_2 \in L^1(\mathbb{R})$.

Another calculation using the convolution theorem, i.e.

$$h := f_1 * f_2 = \mathcal{F}^{-1}(\widehat{f_1} \cdot \widehat{f_2}) = \mathcal{F}^{-1}(g_1 \cdot g_2)$$

leads to

$$h(x) = -\frac{15}{4} \frac{\cos(2\pi x) - 1}{\pi^6 x^6} - 6 \frac{\sin(2\pi x)}{\pi^5 x^5} + 3 \frac{\cos(2\pi x) + \frac{1}{2}}{\pi^4 x^4} + \frac{1/2}{\pi^2 x^2}$$

for $x \in \mathbb{R} \setminus \{0\}$. Since all terms except for the last one decay strictly faster than x^{-2} as $|x| \rightarrow \infty$, we get $|(f_1 * f_2)(x)| = |h(x)| \asymp x^{-2}$ for large $|x|$.

Thus, $f_1 * f_2 \notin L^p(\mathbb{R})$ for $\frac{1}{3} < p \leq \frac{1}{2}$, although $f_1 \in L^1(\mathbb{R})$ and $f_2 \in L^p(\mathbb{R})$ for these values of p . This shows that the usual form of Young's inequality fails for $p < 1$, even if we assume the factors of the convolution to be bandlimited. \blacklozenge

In the example above, one should note that $f_1 * f_2 \in L^p(\mathbb{R})$ holds for $\frac{1}{2} < p < 1$, which is exactly the range of $p \in (0, 1)$, for which $f_1, f_2 \in L^p(\mathbb{R})$ holds. This is indicative of the kind of convolution relation that we are after (cf. also equation (3.2)).

For technical reasons, we start our derivation of the convolution relations for bandlimited L^p -functions with $p \in (0, 1)$ by showing that we can always approximate such functions by Schwartz functions in a suitable way. This will allow us to restrict to Schwartz functions for most of our proofs.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^d$ be compact and assume that $f \in \mathcal{S}'(\mathbb{R}^d)$ is a tempered distribution with compact Fourier support $\widehat{\text{supp}} f \subset \Omega$.*

Then f is given by (integration against) a smooth function $g \in C^\infty(\mathbb{R}^d)$ with polynomially bounded derivatives of all orders.

Furthermore, there is a sequence of Schwartz functions $(g_n)_{n \in \mathbb{N}}$ with the following properties:

(1) $\widehat{\text{supp}} g_n \subset B_{1/n}(\Omega)$, where $B_{1/n}(\Omega)$ is the $\frac{1}{n}$ -neighborhood of Ω , given by

$$B_{1/n}(\Omega) = \left\{ \xi \in \mathbb{R}^d \mid \text{dist}(\xi, \Omega) < \frac{1}{n} \right\}.$$

(2) $|g_n(x)| \leq |g(x)|$ for all $x \in \mathbb{R}^d$ and

(3) $g_n(x) \xrightarrow{n \rightarrow \infty} g(x)$ for all $x \in \mathbb{R}^d$.

In particular, $\|g_n - f\|_{L^p} \xrightarrow{n \rightarrow \infty} 0$ holds for any $p \in (0, \infty)$ for which $f \in L^p(\mathbb{R}^d)$ is true. \blacktriangleleft

Remark. In the following, we will always identify f with its “smooth version” g , i.e. we will write $f(x)$ instead of $g(x)$ for $x \in \mathbb{R}^d$. \blacklozenge

Proof. (based upon the proof of [21, Theorem 1.4.1]) In case of $\Omega = \emptyset$, we have $f = 0$, so that all claims are trivial (with $g_n \equiv 0$). Hence, we can assume $\Omega \neq \emptyset$ in the following. That f is given by (integration against) a smooth (even analytic) function $g \in C^\infty(\mathbb{R}^d)$ with polynomially bounded derivatives of all orders is a consequence of the Paley-Wiener Theorem (cf. [16, Theorem 7.23]). Hence, we only need to establish existence of the sequence $(g_n)_{n \in \mathbb{N}}$.

To this end, choose $\psi \in C_c^\infty(B_1(0))$ with $0 \leq \psi \leq 1$ and $\psi(0) = 1$ and set $\tilde{\varphi} := \mathcal{F}^{-1}\psi \in \mathcal{S}(\mathbb{R}^d)$. Note that

$$\gamma := \tilde{\varphi}(0) = \int_{\mathbb{R}^d} \psi(\xi) \, d\xi > 0,$$

since ψ is continuous and nonnegative with $\psi \not\equiv 0$. Finally, define $\varphi := \tilde{\varphi}/\gamma$ and observe

$$\text{supp } \hat{\varphi} = \text{supp } (\mathcal{F}\tilde{\varphi}) = \text{supp } \psi \subset B_1(0),$$

as well as $\varphi(0) = 1$.

Now, since the Fourier transform $\hat{\varphi} = \psi/\gamma$ is nonnegative, $|\varphi|$ attains its global maximum at the origin, since

$$\begin{aligned} |\varphi(x)| &= |(\mathcal{F}^{-1}\hat{\varphi})(x)| \\ &= \frac{1}{\gamma} |(\mathcal{F}^{-1}\psi)(x)| \\ &\leq \frac{1}{\gamma} \cdot \int_{\mathbb{R}^d} |\psi(\xi) \cdot e^{2\pi i \langle x, \xi \rangle}| \, d\xi \\ &= \frac{1}{\gamma} \cdot \int_{\mathbb{R}^d} \psi(\xi) \, d\xi = \frac{1}{\gamma} \cdot \gamma = 1 \end{aligned}$$

holds for all $x \in \mathbb{R}^d$.

Let us now set

$$g_n : \mathbb{R}^d \rightarrow \mathbb{C}, x \mapsto g(x) \cdot \varphi\left(\frac{x}{n}\right)$$

for $n \in \mathbb{N}$. Since g is a smooth function with polynomially bounded derivatives, the Leibniz rule easily implies that all derivatives of g_n decay rapidly, so that g_n is indeed a Schwartz function. As seen above, $|\varphi(x)| \leq 1$ for all $x \in \mathbb{R}^d$, which immediately yields

$$|g_n(x)| = \left| g(x) \cdot \varphi\left(\frac{x}{n}\right) \right| \leq |g(x)|.$$

Furthermore, $\varphi(0) = 1$ and continuity of φ yield

$$g_n(x) = g(x) \cdot \varphi\left(\frac{x}{n}\right) \xrightarrow{n \rightarrow \infty} g(x) \cdot 1 = g(x).$$

Finally, the convolution theorem (cf. [16, Theorem 7.19(e)] and see [16, Definition 7.18] for the definition of the convolution of a tempered distribution and a Schwartz function) yields

$$\widehat{g_n} = \widehat{g} * \widehat{\varphi\left(\frac{\cdot}{n}\right)} = n^d \cdot [\widehat{g} * (\widehat{\varphi}(n \cdot))].$$

Since $\widehat{\varphi}$ is supported in $B_r(0)$ for some $r \in (0, 1)$, we see that $\widehat{\varphi}(n \cdot)$ has support in $B_{r/n}(0)$.

By definition of convolution for tempered distributions, we have

$$[\widehat{g} * (\widehat{\varphi}(n \cdot))](\xi) = \langle \widehat{g}, L_\xi([\widehat{\varphi}(n \cdot)]^\vee) \rangle_{\mathcal{S}, \mathcal{S}'},$$

with $\theta^\vee(\xi) = \theta(-\xi)$. But since $[\widehat{\varphi}(n \cdot)]^\vee$ is supported in $B_{r/n}(0)$, we see that $L_\xi([\widehat{\varphi}(n \cdot)]^\vee)$ is supported in $B_{r/n}(\xi)$. Thus, $[\widehat{g} * (\widehat{\varphi}(n \cdot))](\xi) \neq 0$ can only happen if $\text{supp } \widehat{g} \cap B_{r/n}(\xi) \neq \emptyset$, i.e. for $\xi \in B_{r/n}(\Omega)$, since $\text{supp } \widehat{g} = \text{supp } \widehat{f} \subset \Omega$. Hence,

$$\text{supp } \widehat{g_n} \subset \overline{B_{r/n}(\Omega)} \subset B_{1/n}(\Omega).$$

The additional claim regarding the L^p -convergence is a direct consequence of the dominated convergence theorem, using the pointwise convergence, together with $|g_n| \leq |g|$ and with $f = g$ almost everywhere if $f \in L^p(\mathbb{R}^d)$. \square

Using the preceding approximation result, we will now show that bandlimited functions which are integrable to a power p are automatically integrable to every power $q \geq p$. This roughly reflects the fact that these functions are always (uniformly) continuous, so that the only obstruction to integrability is the decay at infinity. This embedding into L^q -spaces with larger q will be a central ingredient for the proof of the convolution relations in the Quasi-Banach regime $p \in (0, 1)$.

We remark that the corollary below is a special case of [21, 1.4.1(3)].

Corollary 3.3. *Let $\Omega \subset \mathbb{R}^d$ be compact and assume that $f \in \mathcal{S}'(\mathbb{R}^d)$ is a tempered distribution with compact Fourier support $\widehat{\text{supp}} f \subset \Omega$. Then the following hold:*

(1) *If $f \in L^p(\mathbb{R}^d)$ holds for some $p \in (0, 2]$, then*

$$\|f\|_{L^q} \leq [\lambda(\Omega)]^{\frac{1}{p}-\frac{1}{q}} \cdot \|f\|_{L^p} \quad (3.3)$$

holds for all $q \in [p, \infty]$.

(2) *There is a constant $K = K(\Omega) > 0$ such that*

$$\|f\|_{L^q} \leq K \cdot \|f\|_{L^p} \quad (3.4)$$

holds for all $p \in [1, \infty]$, $q \in [p, \infty]$ and $f \in L^p(\mathbb{R}^d)$ with $\widehat{\text{supp}} f \subset \Omega$. \blacktriangleleft

Proof. We first observe that it suffices to establish the stated estimates for Schwartz functions, since the Fatou property for $L^q(\mathbb{R}^d)$ implies, using the sequence $(g_n)_{n \in \mathbb{N}}$ given by Lemma 3.2, that

$$\begin{aligned} \|f\|_{L^q} &\leq \liminf_{n \rightarrow \infty} \|g_n\|_{L^q} \\ &\leq \liminf_{n \rightarrow \infty} [\lambda(B_{1/n}(\Omega))]^{\frac{1}{p}-\frac{1}{q}} \cdot \|g_n\|_{L^p} \\ &\leq [\lambda(\Omega)]^{\frac{1}{p}-\frac{1}{q}} \cdot \|f\|_{L^p}. \end{aligned}$$

This proves equation (3.3) in the general case. Observe that the last step used $|g_n| \leq |f|$ as well as $\lambda(B_{1/n}(\Omega)) \rightarrow \lambda(\Omega)$. This last convergence is a consequence of the continuity of the (Lebesgue) measure from above (cf. [8, Theorem 1.8(d)]) and of the identity $\Omega = \bigcap_{n \in \mathbb{N}} B_{1/n}(\Omega)$, which holds because Ω is compact.

Validity of equation (3.4) in the general case is derived similarly, with $K = K(\overline{B_1(\Omega)})$ instead of $K = K(\Omega)$.

Hence, we can assume $f \in \mathcal{S}(\mathbb{R}^d)$ with $\widehat{\text{supp}} f \subset \Omega$ in the following. Observe that by regularity of the Lebesgue measure, there is for each $\varepsilon > 0$ some open set $U_\varepsilon \supset \Omega$ with $\lambda(U_\varepsilon) < \lambda(\Omega) + \varepsilon$. Furthermore, there is some $\gamma_\varepsilon \in C_c^\infty(U_\varepsilon)$ with $0 \leq \gamma_\varepsilon \leq 1$ and $\gamma_\varepsilon \equiv 1$ on a neighborhood of Ω . Set $\psi_\varepsilon := \mathcal{F}^{-1}\gamma_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$.

We first observe that the Hausdorff-Young inequality (cf. [8, Theorem 8.30]) yields

$$\|\psi_\varepsilon\|_{L^{p'}} \leq \|\gamma_\varepsilon\|_{L^p} \leq \|\chi_{U_\varepsilon}\|_{L^p} = [\lambda(U_\varepsilon)]^{1/p} \leq [\lambda(\Omega) + \varepsilon]^{1/p} \quad (3.5)$$

for all $p \in [1, 2]$, where $p' \in [2, \infty]$ is the conjugate exponent, i.e. $p' = \frac{p}{p-1}$ for $p \in (1, 2]$ and $p' = \infty$ for $p = 1$.

Using the support assumption on \widehat{f} , we get $\widehat{f} = \widehat{f} \cdot \gamma_\varepsilon$ and hence

$$f = \mathcal{F}^{-1}\widehat{f} = \mathcal{F}^{-1}[\widehat{f} \cdot \gamma_\varepsilon] = f * \psi_\varepsilon.$$

For the proof of equation (3.3), we first note that it suffices to establish the case $q = \infty$. In the remaining case $p \leq q < \infty$, we get

$$\begin{aligned} \|f\|_{L^q}^q &= \int_{\mathbb{R}^d} |f|^p \cdot |f|^{q-p} \, dx \\ &\leq \|f\|_{L^\infty}^{q-p} \cdot \int_{\mathbb{R}^d} |f|^p \, dx \\ &\leq [\lambda(\Omega)]^{\frac{q-p}{p}} \cdot \|f\|_{L^p}^{q-p} \cdot \|f\|_{L^p}^p, \end{aligned}$$

where the last step made use of the case $q = \infty$. Taking q -th roots completes the proof of equation (3.3).

For the case $q = \infty$, we distinguish two sub-cases:

Case 1: If $p \in [1, 2]$, Hölder's inequality implies

$$\begin{aligned} |f(x)| &= |(f * \psi_\varepsilon)(x)| \\ &= \left| \int_{\mathbb{R}^d} f(y) \cdot \psi_\varepsilon(x - y) \, dy \right| \\ &\leq \|f\|_{L^p} \cdot \|\psi_\varepsilon(x - \cdot)\|_{L^{p'}} \\ &\leq [\lambda(\Omega) + \varepsilon]^{1/p} \cdot \|f\|_{L^p} \end{aligned} \tag{3.6}$$

for all $x \in \mathbb{R}^d$. Here, equation (3.5) was used in the last step. Since $\varepsilon > 0$ was arbitrary, this is nothing but equation (3.3) for $q = \infty$ and $p \in [1, 2]$.

Case 2: $p \in (0, 1)$. Here, we cannot use Hölder's inequality as in the last case. Instead, we apply a “flop”. In the present context, this means that we will derive an estimate of the form $Q \leq C \cdot Q^r$ for some exponent $r \in (0, 1)$ with $Q := \sup_{y \in \mathbb{R}^d} |f(y)|$. By rearranging, this yields $Q \leq C^{\frac{1}{1-r}}$, which will imply the desired estimate. Here it is important to note that Q is indeed a finite quantity because of $f \in \mathcal{S}(\mathbb{R}^d)$.

For the execution of this plan, note

$$\begin{aligned} |f(x)| &= \left| \int_{\mathbb{R}^d} f(y) \cdot \psi_\varepsilon(x - y) \, dy \right| \\ &\leq \int_{\mathbb{R}^d} |f(y)|^p \cdot |f(y)|^{1-p} \cdot |\psi_\varepsilon(x - y)| \, dy \\ &\leq \|\psi_\varepsilon\|_{L^\infty} \cdot \|f\|_{L^p}^p \cdot \sup_{y \in \mathbb{R}^d} |f(y)|^{1-p} \\ &\leq (\lambda(\Omega) + \varepsilon) \cdot \|f\|_{L^p}^p \cdot Q^{1-p}. \end{aligned}$$

Here, we again used equation (3.5) (with $p = 1$) in the last step. Furthermore, it is important to observe that we can indeed interchange the supremum and the power $1 - p$ because of $1 - p > 0$.

Taking the supremum over $x \in \mathbb{R}^d$ on the left-hand side yields

$$\sup_{x \in \mathbb{R}^d} |f(x)| \leq (\lambda(\Omega) + \varepsilon) \cdot \|f\|_{L^p}^p \cdot \left(\sup_{x \in \mathbb{R}^d} |f(x)| \right)^{1-p}.$$

Rearranging, taking p -th roots and letting $\varepsilon \downarrow 0$ completes the proof of equation (3.3) for $p \in (0, 1)$ and $q = \infty$.

It remains to establish estimate (3.4). To this end, we simply note that Hölder's inequality implies as in equation (3.6) that

$$|f(x)| \leq \|f\|_{L^p} \cdot \|\psi_1(x - \cdot)\|_{L^{p'}} \leq K \cdot \|f\|_{L^p}$$

with $K := \max\{1, \|\psi_1\|_{L^1}, \|\psi_1\|_{L^\infty}\}$. The last estimate used $\|\psi\|_{L^r} \leq \max\{\|\psi\|_{L^1}, \|\psi\|_{L^\infty}\}$ for all $r \in [1, \infty]$, cf. [8, Proposition 6.10]. This establishes the claim for $q = \infty$.

In general, for $1 \leq p \leq q < \infty$, we have

$$\|f\|_{L^q}^q = \int_{\mathbb{R}^d} |f|^p \cdot |f|^{q-p} \, dx \leq \|f\|_{L^\infty}^{q-p} \cdot \|f\|_{L^p}^p \leq K^{q-p} \cdot \|f\|_{L^p}^{q-p} \cdot \|f\|_{L^p}^p.$$

Rearranging yields $\|f\|_{L^q} \leq K^{1-\frac{p}{q}} \cdot \|f\|_{L^p}$. Because of $K \geq 1$ and $0 \leq 1 - \frac{p}{q} \leq 1$, we have $K^{1-\frac{p}{q}} \leq K$, which completes the proof. \square

Now, we can finally establish the convolution relation for bandlimited functions in $L^p(\mathbb{R}^d)$ for $p \in (0, 1)$. The result is essentially taken from Triebel [21, Proposition 1.5.1]; but Triebel does not state the form of the constant $[\lambda(Q - \Omega)]^{\frac{1}{p}-1}$ explicitly in the statement of the theorem. This constant, however, will be important for us, see e.g. the definition of the weight v (for $q_k < 1$) in Theorem 5.6.

Theorem 3.4. (cf. [21, Proposition 1.5.1]) Let $\Omega, Q \subset \mathbb{R}^d$ be compact and $p \in (0, 1]$. Furthermore, let $\psi \in L^1(\mathbb{R}^d)$ with $\text{supp } \psi \subset Q$ and such that $\mathcal{F}^{-1}\psi \in L^p(\mathbb{R}^d)$.

For each $f \in L^p(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ with Fourier support $\text{supp } \widehat{f} \subset \Omega$, we have

$$\mathcal{F}^{-1}(\psi \cdot \widehat{f}) = (\mathcal{F}^{-1}\psi) * f \in L^p(\mathbb{R}^d)$$

with norm estimate

$$\begin{aligned} \|\mathcal{F}^{-1}(\psi \cdot \widehat{f})\|_{L^p} &= \|(\mathcal{F}^{-1}\psi) * f\|_{L^p} \\ &\leq \| |\mathcal{F}^{-1}\psi| * |f| \|_{L^p} \\ &\leq [\lambda(Q - \Omega)]^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}\psi\|_{L^p} \cdot \|f\|_{L^p}, \end{aligned}$$

where

$$Q - \Omega = \{q - \omega \mid q \in Q \text{ and } \omega \in \Omega\}$$

is the **algebraic difference** of Q and Ω , which is compact and hence measurable.

Finally, we have the pointwise estimate

$$\begin{aligned} |[(\mathcal{F}^{-1}\psi) * f](x)| &\leq \| |\mathcal{F}^{-1}\psi| * |f| \| (x) \\ &\leq [\lambda(Q - \Omega)]^{\frac{1}{p}-1} \left[\int_{\mathbb{R}^d} |(\mathcal{F}^{-1}\psi)(y) \cdot f(x - y)|^p \, dy \right]^{\frac{1}{p}} \end{aligned} \quad (3.7)$$

for all $x \in \mathbb{R}^d$. ◀

Remark. We observe that $\psi \cdot \widehat{f} \in \mathcal{S}'(\mathbb{R}^d)$ is well-defined, even though ψ might not be a smooth function. This is because Corollary 3.3—together with $p \leq 1$ —yields $f \in L^1(\mathbb{R}^d)$ and thus $\widehat{f} \in C_0(\mathbb{R}^d)$. Hence, $\psi \cdot \widehat{f} \in L^1(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ because of $\psi \in L^1(\mathbb{R}^d)$. ◆

Proof. The Riemann-Lebesgue lemma yields $\mathcal{F}^{-1}\psi \in C_0(\mathbb{R}^d)$ because of $\psi \in L^1(\mathbb{R}^d)$. Thus, we get $\mathcal{F}^{-1}\psi \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ for all $q \in [p, \infty]$, cf. [8, Proposition 6.10]. For the same range of q , Corollary 3.3 yields $f \in L^q(\mathbb{R}^d)$.

Because of $p \in (0, 1]$, we get $f, \mathcal{F}^{-1}\psi \in L^1(\mathbb{R}^d)$ and thus also $(\mathcal{F}^{-1}\psi) * f \in L^1(\mathbb{R}^d)$ with Fourier transform

$$\mathcal{F}[(\mathcal{F}^{-1}\psi) * f] = \psi \cdot \widehat{f}.$$

Further, $\psi \in L^1(\mathbb{R}^d)$ together with $\widehat{f} \in C_0(\mathbb{R}^d)$ yields $\psi \cdot \widehat{f} \in L^1(\mathbb{R}^d)$. By Fourier inversion, this implies

$$\mathcal{F}^{-1}(\psi \cdot \widehat{f}) = (\mathcal{F}^{-1}\psi) * f$$

as claimed.

We first observe that the pointwise estimate (3.7) implies the remaining claims. Indeed, equation (3.7), together with Fubini's theorem, yields

$$\begin{aligned} \|(\mathcal{F}^{-1}\psi) * f\|_{L^p}^p &\leq \| |\mathcal{F}^{-1}\psi| * |f| \|_{L^p}^p \\ &\leq [\lambda(Q - \Omega)]^{1-p} \cdot \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(\mathcal{F}^{-1}\psi)(y)|^p \cdot |f(x - y)|^p \, dy \, dx \\ &= [\lambda(Q - \Omega)]^{1-p} \cdot \int_{\mathbb{R}^d} |(\mathcal{F}^{-1}\psi)(y)|^p \cdot \int_{\mathbb{R}^d} |f(x - y)|^p \, dx \, dy \\ &= [\lambda(Q - \Omega)]^{1-p} \cdot \|\mathcal{F}^{-1}\psi\|_{L^p}^p \cdot \|f\|_{L^p}^p, \end{aligned}$$

so that taking p -th roots completes the proof.

In order to prove equation (3.7), fix $x \in \mathbb{R}^d$. We will write $g^\vee(y) = g(-y)$ for arbitrary functions $g : \mathbb{R}^d \rightarrow \mathbb{C}$. Using $f \in L^\infty(\mathbb{R}^d)$ and $\mathcal{F}^{-1}\psi \in L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, we see

$$F_x := (\mathcal{F}^{-1}\psi) \cdot L_x f^\vee \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d),$$

so that the Fourier transform $\mathcal{F}F_x$ is well-defined with

$$\begin{aligned} (\mathcal{F}F_x)(\xi) &= \int_{\mathbb{R}^d} \widehat{\psi}(-y) \cdot f(x-y) \cdot e^{-2\pi i \langle y, \xi \rangle} dy \\ &= \int_{\mathbb{R}^d} \widehat{\psi}(z) \cdot f(x+z) \cdot e^{2\pi i \langle z, \xi \rangle} dz \\ &= \int_{\mathbb{R}^d} \psi(y) \cdot \mathcal{F} \left[z \mapsto f(x+z) \cdot e^{2\pi i \langle z, \xi \rangle} \right] (y) dy, \end{aligned}$$

where the last step used $\psi, f \in L^1(\mathbb{R}^d)$ and that $\int \widehat{f}g = \int f\widehat{g}$ for $f, g \in L^1(\mathbb{R}^d)$, cf. [8, Lemma 8.25].

By elementary properties of the Fourier transform,

$$\mathcal{F} \left[z \mapsto f(x+z) \cdot e^{2\pi i \langle z, \xi \rangle} \right] (y) = \mathcal{F}[M_\xi L_{-x}f](y) = (L_\xi M_x \widehat{f})(y),$$

which yields

$$(\mathcal{F}F_x)(\xi) = \int_{\mathbb{R}^d} \psi(y) \cdot (M_x \widehat{f})(y - \xi) dy = [\psi * (M_x \widehat{f})^\vee](\xi)$$

and hence³

$$\text{supp}(\mathcal{F}F_x) \subset \text{supp} \psi + \text{supp}([M_x \widehat{f}]^\vee) \subset Q + (-\Omega) = Q - \Omega. \quad (3.8)$$

Using Corollary 3.3 and $0 < p \leq 1$, we finally arrive at

$$\begin{aligned} |[(\mathcal{F}^{-1}\psi) * f](x)| &\leq (|\mathcal{F}^{-1}\psi| * |f|)(x) \\ &= \int_{\mathbb{R}^d} |(\mathcal{F}^{-1}\psi)(y) \cdot f(x-y)| dy \\ &= \|F_x\|_{L^1} \leq [\lambda(Q - \Omega)]^{\frac{1}{p}-\frac{1}{q}} \cdot \|F_x\|_{L^p}, \end{aligned}$$

which is nothing but the pointwise estimate (3.7). \square

3.2. Definition of decomposition spaces. In this subsection, we will finally define the (Fourier side) decomposition space $\mathcal{D}_{\mathcal{F}}(Q, L^p, Y)$ and its “space side” version. To this end, we first introduce the class of partitions of unity which will turn out to be suitable for the definition of decomposition spaces. We begin with the case $p \in [1, \infty]$, since we will have to place more restrictive assumptions on the covering Q in the Quasi-Banach regime $p \in (0, 1)$, cf. Definition 3.6.

Definition 3.5. (cf. [7, Definition 2.2])

Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open and let $Q = (Q_i)_{i \in I}$ be an admissible covering of \mathcal{O} . A family $\Phi = (\varphi_i)_{i \in I}$ of functions on \mathcal{O} is called an **L^p -bounded admissible partition of unity (L^p -BAPU)** for Q for all $1 \leq p \leq \infty$, if

- (1) $\varphi_i \in C_c^\infty(\mathcal{O})$ for all $i \in I$,
- (2) $\sum_{i \in I} \varphi_i(\xi) = 1$ for all $\xi \in \mathcal{O}$,
- (3) $\varphi_i(\xi) = 0$ for all $\xi \in \mathbb{R}^d \setminus Q_i$ for all $i \in I$, and
- (4) the constant $C_{Q, \Phi, p} := \sup_{i \in I} \|\mathcal{F}^{-1}\varphi_i\|_{L^1}$ is finite.

We say that an admissible covering Q of \mathcal{O} is an **L^p -decomposition covering** of \mathcal{O} for all $1 \leq p \leq \infty$ if there is an L^p -BAPU Φ for Q . \blacktriangleleft

Remark. The term “ L^p -bounded” used above does *not* refer to the fact that the L^p -norm of the φ_i is uniformly bounded, but to the fact that $(\varphi_i)_{i \in I}$ forms a uniformly bounded family of L^p Fourier-multipliers, as a consequence of Young’s inequality $L^1 * L^p \hookrightarrow L^p$.

Clearly, the constant $C_{Q, \Phi, p}$ does *not* actually depend on Q and p ; but below, we will introduce the concept of an L^p -BAPU also for $p \in (0, 1)$ and in this case, the similarly defined constant will depend on Q and p . For consistency, we write $C_{Q, \Phi, p}$ also for $p \in [1, \infty]$.

Finally, we mention that the term “BAPU” goes back to Feichtinger and Gröbner[7]. \blacklozenge

³Note that the distribution \widehat{f} is assumed to satisfy $\text{supp} \widehat{f} \subset \Omega$. Because of $f \in L^1(\mathbb{R}^d)$, \widehat{f} is given by integration against a continuous bounded function. It is then easy to see that this continuous function also has support in Ω . This justifies the calculation in equation (3.8).

For the Quasi-Banach regime $p \in (0, 1)$, the following definition will turn out to be suitable. Note that we assume the covering \mathcal{Q} to be semi-structured, whereas for $p \in [1, \infty]$ all we needed was an admissible covering.

Definition 3.6. (cf. [4, Definition 2])

Let $0 < p < 1$, let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open and assume that $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ is a semi-structured covering of \mathcal{O} . We say that a family $\Phi = (\varphi_i)_{i \in I}$ is an **L^p -bounded admissible partition of unity** (**L^p -BAPU**) for \mathcal{Q} , if

- (1) $\varphi_i \in C_c^\infty(\mathcal{O})$ for all $i \in I$,
- (2) $\sum_{i \in I} \varphi_i(\xi) = 1$ for all $\xi \in \mathcal{O}$,
- (3) $\varphi_i(\xi) = 0$ for all $\xi \in \mathbb{R}^d \setminus Q_i$ for all $i \in I$, and
- (4) the constant

$$C_{\mathcal{Q}, \Phi, p} := \sup_{i \in I} \left(|\det T_i|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1} \varphi_i\|_{L^p} \right)$$

is finite.

We say that \mathcal{Q} is an **L^p -decomposition covering** of \mathcal{O} if there is an L^p -BAPU Φ for \mathcal{Q} . ◀

Remark. We will see in Corollary 5.4 that every L^p -BAPU for \mathcal{Q} is automatically an L^q -BAPU for \mathcal{Q} for all $q \in [p, \infty]$. ♦

Before we finally give a formal definition of the decomposition space $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$, we first clarify our assumptions on the space Y .

Definition 3.7. (cf. [7, Definition 2.5])

- (1) Let I be an index set. We say that a quasi-normed vector space $(Y, \|\cdot\|_Y)$ is a **solid sequence space over I** if the following hold:
 - (a) Y is a subspace of \mathbb{C}^I ,
 - (b) If $x = (x_i)_{i \in I} \in \mathbb{C}^I$ and $y = (y_i)_{i \in I} \in Y$ with $|x_i| \leq |y_i|$ for all $i \in I$, then $x \in Y$ and $\|x\|_Y \leq \|y\|_Y$.
- (2) Let $\mathcal{Q} = (Q_i)_{i \in I}$ be a covering of a set $X \neq \emptyset$. We say that solid sequence space $(Y, \|\cdot\|_Y)$ over I is **\mathcal{Q} -regular**, if the following hold:
 - (a) Y is complete, i.e. a Quasi-Banach space,
 - (b) Y is **invariant under \mathcal{Q} -clustering**, i.e., the **\mathcal{Q} -clustering map**

$$\Gamma_{\mathcal{Q}} : Y \rightarrow Y, (c_i)_{i \in I} \mapsto (c_i^*)_{i \in I} := \left(\sum_{j \in i^*} c_j \right)_{i \in I}$$

is well-defined and bounded. ◀

Remark. If all of the sets i^* are finite (e.g. if \mathcal{Q} is admissible), then the closed graph theorem (cf. [16, Theorem 2.15]), together with completeness of Y and with the continuous embedding $Y \hookrightarrow \mathbb{C}^I$ (which follows from Lemma 3.8 below) and the fact that the clustering map is continuous with respect to the (Hausdorff!) product topology on \mathbb{C}^I , imply that the clustering map $\Gamma_{\mathcal{Q}}$ is bounded iff it is well-defined. ♦

Lemma 3.8. Let $I \neq \emptyset$ be a set and let $(Y, \|\cdot\|_Y)$ be a solid sequence space over I . For $i \in I$, set

$$u_i := \begin{cases} \|\delta_i\|_Y, & \text{if } \delta_i \in Y, \\ 1, & \text{if } \delta_i \notin Y, \end{cases}$$

where $(\delta_i)_j = 0$ for $j \neq i$ and $(\delta_i)_i = 1$.

Then $Y \hookrightarrow \ell_u^\infty(I)$. More precisely, we even have $\|x\|_{\ell_u^\infty} \leq \|x\|_Y$ for all $x = (x_i)_{i \in I} \in Y$. ◀

Proof. Let $x = (x_i)_{i \in I} \in Y$ be arbitrary and let $i \in I$. In case of $x_i \neq 0$, we have

$$|(\delta_i)_j| \leq \frac{|x_j|}{|x_i|} \text{ for all } j \in I.$$

By solidity of Y , this implies $\delta_i \in Y$ with

$$\|\delta_i\|_Y \leq \left\| \frac{x}{|x_i|} \right\|_Y = \frac{\|x\|_Y}{|x_i|},$$

whence

$$u_i |x_i| = \|\delta_i\|_Y \cdot |x_i| \leq \|x\|_Y.$$

In case of $x_i = 0$, we trivially have $u_i \cdot |x_i| = 0 \leq \|x\|_Y$.

Since $i \in I$ was arbitrary, we conclude $\|x\|_{\ell_u^\infty} \leq \|x\|_Y < \infty$, which completes the proof. \square

The most important example of \mathcal{Q} -regular sequence spaces that we will consider are weighted ℓ^q -spaces. As we will see in Lemma 4.13, $\ell_u^q(I)$ is \mathcal{Q} -regular, whenever $u = (u_i)_{i \in I}$ is \mathcal{Q} -moderate.

For later use, we state the following result which connects iterated applications of the clustering map and summing over the clustered index sets i^* .

Lemma 3.9. *Let $\mathcal{Q} = (Q_i)_{i \in I}$ be an admissible covering of a set $X \neq \emptyset$ and let $\Gamma_{\mathcal{Q}} : \mathbb{C}^I \rightarrow \mathbb{C}^I, c \mapsto c^*$ denote the \mathcal{Q} -clustering map.*

For each sequence $c = (c_i)_{i \in I}$ with nonnegative terms $c_i \geq 0$ and each $\ell \in \mathbb{N}$, we have

$$(\Gamma_{\mathcal{Q}}^\ell c)_i \geq \sum_{j \in i^{\ell*}} c_j \quad \text{for all } i \in I. \quad (3.9)$$

In particular, if Y is \mathcal{Q} -regular, then the ℓ -fold clustering map $\Theta_\ell : Y \rightarrow Y$ with

$$(\Theta_\ell c)_i := \sum_{j \in i^{\ell*}} c_j$$

is well-defined and bounded with $\|\Theta_\ell\|_{Y \rightarrow Y} \leq \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}^\ell$. \blacktriangleleft

Proof. We first prove estimate (3.9) by induction on $\ell \in \mathbb{N}$. For $\ell = 1$, we have equality by definition of $\Gamma_{\mathcal{Q}}$.

For the induction step, fix for each $j \in i^{(\ell+1)*}$ some $k_j \in i^{\ell*}$ with $j \in k_j^*$. Since each term c_j is nonnegative, this yields

$$\begin{aligned} \sum_{j \in i^{(\ell+1)*}} c_j &= \sum_{k \in i^{\ell*}} \sum_{\substack{j \in i^{(\ell+1)*} \\ \text{with } k_j = k}} c_j \\ (j \in k_j^* = k^* \text{ if } k_j = k) &\leq \sum_{k \in i^{\ell*}} \sum_{j \in k^*} c_j \\ &= \sum_{k \in i^{\ell*}} (\Gamma_{\mathcal{Q}} c)_k \\ &\stackrel{(*)}{\leq} (\Gamma_{\mathcal{Q}}^\ell \Gamma_{\mathcal{Q}} c)_i = (\Gamma_{\mathcal{Q}}^{\ell+1} c)_i, \end{aligned}$$

where we employed the induction hypothesis (with the nonnegative(!) sequence $\Gamma_{\mathcal{Q}} c$ instead of c) at $(*)$.

To prove boundedness of Θ_ℓ , let $c = (c_i)_{i \in I} \in Y$ be arbitrary and set $d := (|c_i|)_{i \in I}$. Observe $d \in Y$ with $\|d\|_Y = \|c\|_Y$, since Y is solid. As seen above, we have

$$|(\Theta_\ell c)_i| \leq \sum_{j \in i^{\ell*}} |c_j| \leq (\Gamma_{\mathcal{Q}}^\ell d)_i$$

for all $i \in I$. Because of $\Gamma_{\mathcal{Q}}^\ell d \in Y$, solidity of Y implies $\Theta_\ell c \in Y$ with

$$\|\Theta_\ell c\|_Y \leq \|\Gamma_{\mathcal{Q}}^\ell d\|_Y \leq \|\Gamma_{\mathcal{Q}}^\ell\|_{Y \rightarrow Y} \|d\|_Y \leq \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}^\ell \cdot \|c\|_Y < \infty. \quad \square$$

Now, we are in a position to define the (Fourier-side) decomposition spaces.

Definition 3.10. Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be an open set and let $p \in (0, \infty]$. Assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an L^p -decomposition covering of \mathcal{O} with L^p -BAPU $\Phi = (\varphi_i)_{i \in I}$ and that $Y \leq \mathbb{C}^I$ is \mathcal{Q} -regular.

For $f \in \mathcal{D}'(\mathcal{O})$, define

$$\|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} := \|f\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)} := \left\| \left(\|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p} \right)_{i \in I} \right\|_Y \in [0, \infty],$$

with the convention that for a family $c = (c_i)_{i \in I}$ with $c_i \in [0, \infty]$, the expression $\|c\|_Y$ is to be read as ∞ if $c_i = \infty$ for some $i \in I$ or if $c \notin Y$.

Define the **(Fourier-side) decomposition space** $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ with respect to the covering \mathcal{Q} , integrability exponent p and global component Y as

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y) := \left\{ f \in \mathcal{D}'(\mathcal{O}) \mid \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} < \infty \right\}. \quad \blacktriangleleft$$

Remark. We remark that $\varphi_i f$ is a distribution on \mathcal{O} with compact support, which thus extends to a (tempered) distribution on \mathbb{R}^d . By Lemma 3.2, this implies that $\mathcal{F}^{-1}(\varphi_i f) \in \mathcal{S}'(\mathbb{R}^d)$ is given by (integration against) a smooth function. Thus, it makes sense to write $\|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p}$, with the caveat that this expression could be infinite.

We finally remark that the notations $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)}$ and $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ suppress the family $(\varphi_i)_{i \in I}$ used to define the (quasi)-norm above. We will see below (cf. Theorem 3.18) that the resulting space is independent of the chosen L^p -BAPU, with equivalent quasi-norms for different choices, so that this is justified. \blacklozenge

For completeness, we also define “space-side” decomposition spaces. To this end, we first introduce the reservoir $Z'(\mathcal{O})$ which we will use for these spaces.

Definition 3.11. For $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ open, we define

$$Z(\mathcal{O}) := \mathcal{F}(C_c^\infty(\mathcal{O})) := \{\hat{f} \mid f \in C_c^\infty(\mathcal{O})\} \leq \mathcal{S}(\mathbb{R}^d)$$

and endow this space with the unique topology that makes the Fourier transform

$$\mathcal{F} : C_c^\infty(\mathcal{O}) \rightarrow Z(\mathcal{O})$$

a homeomorphism.

We equip the topological dual space $Z'(\mathcal{O}) := [Z(\mathcal{O})]'$ of $Z(\mathcal{O})$ with the weak- $*$ -topology, i.e., with the topology of pointwise convergence on $Z(\mathcal{O})$.

Finally, as on the Schwartz space, we extend the Fourier transform by duality to $Z'(\mathcal{O})$, i.e. we define

$$\mathcal{F} : Z'(\mathcal{O}) \rightarrow \mathcal{D}'(\mathcal{O}), f \mapsto f \circ \mathcal{F}. \quad (3.10)$$

As usual, we write $\hat{f} := \mathcal{F}f$ for $f \in Z'(\mathcal{O})$. \blacktriangleleft

Remark. Since $\mathcal{F} : C_c^\infty(\mathcal{O}) \rightarrow Z(\mathcal{O})$ is a linear homeomorphism, the Fourier transform as defined in equation (3.10) is easily seen to be a homeomorphism. \blacklozenge

Now, we are in a position to define the *space-side* decomposition spaces.

Definition 3.12. Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be an open set and let $p \in (0, \infty]$. Assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an L^p -decomposition covering of \mathcal{O} with L^p -BAPU $(\varphi_i)_{i \in I}$ and that $Y \leq \mathbb{C}^I$ is \mathcal{Q} -regular.

For $f \in Z'(\mathcal{O})$, set

$$\|f\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)} := \|\hat{f}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} = \left\| \left(\|\mathcal{F}^{-1}(\varphi_i \hat{f})\|_{L^p} \right)_{i \in I} \right\|_Y \in [0, \infty]$$

and define the **space-side decomposition space** $\mathcal{D}(\mathcal{Q}, L^p, Y)$ with respect to the covering \mathcal{Q} , integrability exponent p and global component Y by

$$\mathcal{D}(\mathcal{Q}, L^p, Y) := \left\{ f \in Z'(\mathcal{O}) \mid \|f\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)} < \infty \right\}. \quad \blacktriangleleft$$

Remark 3.13. Since the Fourier transform $\mathcal{F} : Z'(\mathcal{O}) \rightarrow \mathcal{D}'(\mathcal{O})$ is an isomorphism, it is clear that the Fourier transform restricts to an (isometric) isomorphism

$$\mathcal{F} : \mathcal{D}(\mathcal{Q}, L^p, Y) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y).$$

Hence, for most purposes, it does not matter whether one considers the “space-side” or the “Fourier-side” version of these spaces. But since the space $\mathcal{D}'(\mathcal{O})$ is a widely known standard space, whereas $Z'(\mathcal{O})$ is not, we prefer to work with the “Fourier-side” spaces. Note in particular that most working analysts have acquired significant intuition on which operations (like differentiation and multiplication with functions from $C^\infty(\mathcal{O})$) are permitted on elements of $\mathcal{D}'(\mathcal{O})$, while this is not true for the space $Z'(\mathcal{O})$.

The main reasons for using the space $\mathcal{D}'(\mathcal{O})$ of distributions on \mathcal{O} instead of the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions are the following:

- (1) We want to allow the case $\mathcal{O} \subsetneq \mathbb{R}^d$. If we were to use the space $\mathcal{S}'(\mathbb{R}^d)$, the decomposition space (quasi)-norm would *not* be positive definite, or we would have to factor out a certain subspace of $\mathcal{S}'(\mathbb{R}^d)$. This is for example done in the usual definition of *homogeneous* Besov spaces, which are subspaces of $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}$, where \mathcal{P} is the space of polynomials. Here, it seems more natural to use the space $\mathcal{D}'(\mathcal{O})$.
- (2) In case of $\mathcal{O} = \mathbb{R}^d$, one could use $\mathcal{S}'(\mathbb{R}^d)$ as the reservoir. For example, Borup and Nielsen[4] define their decomposition spaces as

$$\mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^p, Y) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \mid \|f\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)} < \infty \right\}.$$

But as we will see below (cf. Example 3.22), this does in general *not* yield a complete space, even for the uniform covering \mathcal{Q} of \mathbb{R}^d and $Y = \ell_u^1$ with a \mathcal{Q} -moderate weight u .

Nevertheless, in Section 8, we will develop criteria which yield the continuous embedding $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$. If these conditions are satisfied, it is easy to see (at least for $\mathcal{O} = \mathbb{R}^d$) that the space $\mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^p, Y)$ from above is complete, since it coincides (up to trivial identifications) with $\mathcal{D}(\mathcal{Q}, L^p, Y)$.

We finally remark that the notations $Z(\mathcal{O})$ and $Z'(\mathcal{O})$ are inspired by Triebel's book [20], cf. in particular [20, Sections 2.2.1-2.2.4]. Triebel also defines spaces very similar to $\mathcal{D}(\mathcal{Q}, L^p, Y)$, but restricts to the case in which the covering \mathcal{Q} consists of (closed) rectangles with sides parallel to the coordinate axes. This is due to the fact that he also considers spaces of Triebel-Lizorkin type as opposed to the spaces of Besov type that we consider here. \blacklozenge

3.3. Well-definedness of decomposition spaces. Our goal in this subsection is to show that the decomposition space $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ is independent of the chosen L^p -BAPU $(\varphi_i)_{i \in I}$ (for \mathcal{Q} -regular global components Y). For later use, we will actually show a slightly stronger statement, namely that one can use an arbitrary L^p -**bounded control system** (defined below) to obtain an equivalent (quasi)-norm.

The proofs of these results depend crucially on certain facts about Fourier multipliers. For the range $p \in [1, \infty]$, Young's inequality will be sufficient. But in the Quasi-Banach regime $p \in (0, 1)$, we have to resort to Theorem 3.4. To make application of this theorem more convenient, we note the following special case.

Corollary 3.14. *Let $p_0 \in (0, 1]$ and let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ be an L^{p_0} -decomposition covering of the open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$.*

For each $n \in \mathbb{N}_0$, there is a constant $C = C(\mathcal{Q}, n, d, p_0) > 0$ with the following property: If $p \in [p_0, 1]$ and $i \in I$ and furthermore

- (1) $f \in L^1(\mathbb{R}^d)$ with $\text{supp } f \subset \overline{Q_i^{n*}}$ and $\mathcal{F}^{-1}f \in L^p(\mathbb{R}^d)$,
- (2) and $g \in \mathcal{D}'(\mathcal{O})$ with $\text{supp } g \subset \overline{Q_i^{n*}}$ and $\mathcal{F}^{-1}g \in L^p(\mathbb{R}^d)$,

then $\mathcal{F}^{-1}(fg) \in L^p(\mathbb{R}^d)$ with

$$\|\mathcal{F}^{-1}(fg)\|_{L^p} \leq C \cdot |\det T_i|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}f\|_{L^p} \cdot \|\mathcal{F}^{-1}g\|_{L^p}. \quad \blacktriangleleft$$

Proof. Lemma 2.4 implies that $\overline{Q_i^{n*}} \subset \bigcup_{j \in i^{n*}} \overline{Q_j} \subset \mathcal{O}$ is compact, since i^{n*} is finite. Hence, $g \in \mathcal{D}'(\mathcal{O})$ is a distribution with compact support in \mathcal{O} , which means that g extends to a tempered distribution $g \in \mathcal{S}'(\mathbb{R}^d)$, cf. [16, Theorem 6.24(d) and Example 7.12(a)]. Hence, we have $\mathcal{F}^{-1}g \in \mathcal{S}'(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $\widehat{\text{supp } \mathcal{F}^{-1}g} = \text{supp } g \subset \overline{Q_i^{n*}}$. By Theorem 3.4, this implies

$$\mathcal{F}^{-1}(fg) = \mathcal{F}^{-1}\left(f \cdot \widehat{\mathcal{F}^{-1}g}\right) \in L^p(\mathbb{R}^d)$$

with

$$\begin{aligned} \|\mathcal{F}^{-1}(fg)\|_{L^p} &= \left\| \mathcal{F}^{-1}\left(f \cdot \widehat{\mathcal{F}^{-1}g}\right) \right\|_{L^p} \\ &\leq [\lambda(\overline{Q_i^{n*}} - \overline{Q_i^{n*}})]^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}f\|_{L^p} \cdot \|\mathcal{F}^{-1}g\|_{L^p}. \end{aligned}$$

But Corollary 2.8 yields a constant $C = C(\mathcal{Q}, n, d) > 0$ with

$$\lambda(\overline{Q_i^{n*}} - \overline{Q_i^{n*}}) \leq C \cdot |\det T_i| \quad \text{for all } i \in I.$$

Without loss of generality, $C \geq 1$. Recall $0 < p_0 \leq p \leq 1$ and hence $0 \leq \frac{1}{p} - 1 \leq \frac{1}{p} \leq \frac{1}{p_0}$. All in all, we conclude

$$\begin{aligned} \|\mathcal{F}^{-1}(fg)\|_{L^p} &\leq [\lambda(\overline{Q_i^{n*}} - \overline{Q_i^{n*}})]^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}f\|_{L^p} \cdot \|\mathcal{F}^{-1}g\|_{L^p} \\ &\leq C^{\frac{1}{p}-1} \cdot |\det T_i|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}f\|_{L^p} \cdot \|\mathcal{F}^{-1}g\|_{L^p} \\ &\text{(since } C \geq 1) \leq C^{1/p_0} \cdot |\det T_i|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}f\|_{L^p} \cdot \|\mathcal{F}^{-1}g\|_{L^p}, \end{aligned}$$

which completes the proof. \square

As noted above, instead of just proving independence of $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ of the chosen L^p -BAPU, we will establish a slightly stronger claim which will use the notion of an L^p -**bounded (control) system**. This concept is based on that of a **bounded control system** as introduced in [7, Definition 2.6].

Definition 3.15. Let $p \in (0, \infty]$, let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open and let $\mathcal{Q} = (Q_i)_{i \in I}$ be an admissible covering of \mathcal{O} . For $p \in (0, 1)$, assume additionally that $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ is semi-structured.

A family $\Gamma = (\gamma_i)_{i \in I}$ of functions $\gamma_i \in C_c^\infty(\mathcal{O})$ is called an L^p -**bounded system** for \mathcal{Q} if the following conditions are satisfied:

- (1) There is some $\ell = \ell_{\Gamma, \mathcal{Q}} \in \mathbb{N}_0$ with $\gamma_i \equiv 0$ on $\mathbb{R}^d \setminus Q_i^{\ell*}$ for all $i \in I$.
- (2) The constant

$$C_{\mathcal{Q}, \Gamma, p} := \begin{cases} \sup_{i \in I} \|\mathcal{F}^{-1}\gamma_i\|_{L^1}, & \text{for } p \in [1, \infty], \\ \sup_{i \in I} |\det T_i|^{\frac{1}{p}-1} \|\mathcal{F}^{-1}\gamma_i\|_{L^p}, & \text{for } p \in (0, 1) \end{cases}$$

is finite.

If furthermore $\gamma_i \equiv 1$ on Q_i for all $i \in I$, we say that Γ is an L^p -**bounded control system** for \mathcal{Q} . \blacktriangleleft

Remark. We remark once more that the terminology “ L^p -bounded” does *not* refer to the fact that the $\mathcal{F}L^p$ -norms $\|\mathcal{F}^{-1}\gamma_i\|_{L^p}$ are bounded, but to the fact that the $(\gamma_i)_{i \in I}$ form a uniformly bounded family of L^p -Fourier multipliers, at least for $p \in [1, \infty]$. For $p \in (0, 1)$, this is only true restricted to the space of L^p -functions with Fourier support near Q_i , cf. Corollary 3.14. \blacklozenge

Remark 3.16. If $\ell \in \mathbb{N}_0$ is fixed and for each $i \in I$, some set $M_i \subset i^{\ell*}$ is selected, then the family $\Gamma = (\varphi_{M_i})_{i \in I}$ is an L^p -bounded system for \mathcal{Q} if $(\varphi_i)_{i \in I}$ is an L^p -BAPU for \mathcal{Q} . To see this, distinguish two cases:

Case 1: If $p \in [1, \infty]$, simply note (using the triangle inequality for L^1) that

$$\|\mathcal{F}^{-1}\gamma_i\|_{L^1} \leq \sum_{j \in M_i} \|\mathcal{F}^{-1}\varphi_j\|_{L^1} \leq |M_i| \cdot C_{\mathcal{Q}, \Phi, p} \leq |i^{\ell*}| \cdot C_{\mathcal{Q}, \Phi, p} \leq N_{\mathcal{Q}}^\ell C_{\mathcal{Q}, \Phi, p}.$$

Case 2: For $p \in (0, 1)$, we first note that the weight $(|\det T_i|)_{i \in I}$ is \mathcal{Q} -moderate. Indeed, Hadamard’s inequality implies $|\det A| \leq \|A\|^d$ for all $A \in \mathbb{R}^{d \times d}$. In particular,

$$\sup_{i \in I} \sup_{j \in i^*} \frac{|\det T_j|}{|\det T_i|} = \sup_{i \in I} \sup_{j \in i^*} |\det(T_i^{-1}T_j)| \leq \sup_{i \in I} \sup_{j \in i^*} \|T_i^{-1}T_j\|^d \leq C_{\mathcal{Q}}^d. \quad (3.11)$$

Now, using the estimate $\left\| \sum_{j=1}^n f_j \right\|_{L^p} \leq n^{\frac{1}{p}-1} \cdot \sum_{j=1}^n \|f_j\|_{L^p}$ (cf. [10, Exercise 1.1.5(d)]) and the bound $|M_i| \leq |i^{\ell*}| \leq N_{\mathcal{Q}}^\ell$, we conclude

$$\begin{aligned} |\det T_i|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p} &\leq N_{\mathcal{Q}}^{\ell(\frac{1}{p}-1)} \cdot \sum_{j \in i^{\ell*}} |\det T_i|^{\frac{1}{p}-1} \|\mathcal{F}^{-1}\varphi_j\|_{L^p} \\ &\leq N_{\mathcal{Q}}^{\ell(\frac{1}{p}-1)} \cdot \sum_{j \in i^{\ell*}} \left[C_{\mathcal{Q}}^{d(\frac{1}{p}-1)} \cdot |\det T_j|^{\frac{1}{p}-1} \|\mathcal{F}^{-1}\varphi_j\|_{L^p} \right] \\ &\leq N_{\mathcal{Q}}^{\ell(\frac{1}{p}-1)} C_{\mathcal{Q}}^{d(\frac{1}{p}-1)} \cdot |i^{\ell*}| \cdot C_{\mathcal{Q}, \Phi, p} \\ &\leq N_{\mathcal{Q}}^{\ell/p} \cdot C_{\mathcal{Q}}^{d(\frac{1}{p}-1)} \cdot C_{\mathcal{Q}, \Phi, p} \end{aligned}$$

for all $i \in I$.

All in all, this yields (for both cases) an estimate of the form $C_{\mathcal{Q},\Gamma,p} \leq C(\mathcal{Q}, C_{\mathcal{Q},\Phi,p}, d, p, \ell)$.

Finally, if $M_i \supset i^*$ holds for all $i \in I$, then Lemma 2.4 shows $\varphi_{M_i} \equiv 1$ on Q_i for all $i \in I$, so that Γ is an L^p -bounded *control* system for \mathcal{Q} . \blacklozenge

With these notions, we can now state the following result which allows to use an L^p -bounded (control) system instead of an L^p -BAPU to determine the quasi-norm on $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ (up to constant factors). We remark that the statement of the theorem is essentially the same as [7, Corollary 2.5], but the case $p \in (0, 1)$ is of course not covered by the setting considered in [7]. Roughly, the theorem shows the following:

- (1) Instead of a \mathcal{Q} -BAPU $\Phi = (\varphi_i)_{i \in I}$, one can use any L^p -bounded control system $\Gamma = (\gamma_i)_{i \in I}$ to calculate the (quasi)-norm $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)}$. Essentially, this means that instead of requiring $\text{supp } \varphi_i \subset Q_i$, it is enough to require $\text{supp } \gamma_i \subset Q_i^{k*}$ for a fixed k . Likewise, instead of $\sum_{i \in I} \varphi_i \equiv 1$ on \mathcal{Q} , it suffices if $\gamma_i \equiv 1$ on Q_i for each $i \in I$.
- (2) Given a (not necessarily disjoint) “partition” $I = \bigcup_{r=1}^{r_0} I^{(r)}$ of the index set I , we can calculate the (quasi)-norm $\|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)}$ “localized” to each of the sets $Q^{(r)} = \bigcup_{i \in I^{(r)}} Q_i$ and then aggregate the individual contributions. This will become useful in connection with the disjointization lemma (Lemma 2.14).

Regarding the notation in the following theorem, if $Y \leq \mathbb{C}^I$ is a solid sequence space on I and $J \subset I$ is an arbitrary subset, we define the **restricted sequence space** $Y|_J$ as

$$Y|_J := \{c = (c_\ell)_{\ell \in J} \in \mathbb{C}^J \mid \tilde{c} \in Y\}, \quad (3.12)$$

where $\tilde{c} = (c_i)_{i \in I}$ is the **trivial extension** of $c = (c_\ell)_{\ell \in J}$ onto I , i.e. $c_i = 0$ for $i \in I \setminus J$. As expected, we let $\|c\|_{Y|_J} := \|\tilde{c}\|_Y$. With this definition, $Y|_J \leq \mathbb{C}^J$ is a solid sequence space on J . Furthermore, if Y is complete, then so is $Y|_J$, since $c \mapsto \tilde{c}$ defines an isometric isomorphism between $Y|_J$ and a closed subspace of Y . Closedness of this subspace is a consequence of Lemma 3.8.

Theorem 3.17. Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open, let $\mathcal{Q} = (Q_i)_{i \in I}$ be an L^p -decomposition covering of \mathcal{O} for some $p \in (0, \infty]$ and let $Y \leq \mathbb{C}^I$ be \mathcal{Q} -regular. Furthermore, let $r_0 \in \mathbb{N}$ and assume $I = \bigcup_{r=1}^{r_0} I^{(r)}$ for certain subsets $I^{(r)} \subset I$. Finally, assume that $\Gamma = (\gamma_i)_{i \in I}$ is an L^p -bounded system for \mathcal{Q} .

For $f \in \mathcal{D}'(\mathcal{O})$, define

$$\|f\|_{\Gamma, (I^{(r)})_r, L^p, Y} := \sum_{r=1}^{r_0} \left\| \left(\|\mathcal{F}^{-1}(\gamma_i f)\|_{L^p} \right)_{i \in I^{(r)}} \right\|_{Y|_{I^{(r)}}} \in [0, \infty].$$

If $\Gamma_{\mathcal{Q}} : Y \rightarrow Y, c \mapsto c^*$ denotes the clustering map, then there is a positive constant

$$\begin{cases} C(\mathcal{Q}, p, r_0, d, \ell_{\Gamma, \mathcal{Q}}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}), & \text{if } p \in (0, 1), \\ C(N_{\mathcal{Q}}, p, r_0, \ell_{\Gamma, \mathcal{Q}}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}), & \text{if } p \in [1, \infty] \end{cases}$$

with

$$\|f\|_{\Gamma, (I^{(r)})_r, L^p, Y} \leq C \cdot C_{\mathcal{Q}, \Gamma, p} \cdot \|f\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)}$$

for all $f \in \mathcal{D}'(\mathcal{O})$ and each L^p -BAPU $\Phi = (\varphi_i)_{i \in I}$ for \mathcal{Q} .

Conversely, if Γ is an L^p -bounded *control* system for \mathcal{Q} and if $C_0 \geq 1$ denotes a triangle constant for Y , then there is a constant

$$C' = \begin{cases} C'(\mathcal{Q}, p, d, C_0, r_0, \ell_{\Gamma, \mathcal{Q}}), & \text{if } p \in (0, 1), \\ C'(C_0, r_0), & \text{if } p \in [1, \infty] \end{cases}$$

with

$$\|f\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)} \leq C' \cdot C_{\mathcal{Q}, \Phi, p} \cdot \|f\|_{\Gamma, (I^{(r)})_r, L^p, Y} \text{ for all } f \in \mathcal{D}'(\mathcal{O})$$

for each L^p -BAPU $\Phi = (\varphi_i)_{i \in I}$ for \mathcal{Q} . \blacktriangleleft

Proof. Let $\ell := \ell_{\Gamma, \mathcal{Q}}$. Let $\Phi = (\varphi_i)_{i \in I}$ be an L^p -BAPU for \mathcal{Q} . Using Lemma 2.4, we get $\varphi_i^{(\ell+1)*} \equiv 1$ on $Q_i^{\ell*}$ for all $i \in I$. Because of $\gamma_i \equiv 0$ on $\mathbb{R}^d \setminus Q_i^{\ell*}$, we conclude

$$\gamma_i = \varphi_i^{(\ell+1)*} \gamma_i = \sum_{j \in i^{(\ell+1)*}} \gamma_i \varphi_j.$$

We begin by proving the first estimate. To this end, let $f \in \mathcal{D}'(\mathcal{O})$ with $\|f\|_{\mathcal{D}_{\mathcal{F},\Phi}(\mathcal{Q},L^p,Y)} < \infty$ (otherwise, the estimate is trivial). Since $L^p(\mathbb{R}^d)$ is a quasi-normed vector space and since Lemma 2.9 yields the uniform bound $|i^{(\ell+1)*}| \leq N_{\mathcal{Q}}^{\ell+1}$, there is a constant $C_1 = C_1(N_{\mathcal{Q}},\ell,p) > 0$ with

$$\begin{aligned} \|\mathcal{F}^{-1}(\gamma_i f)\|_{L^p} &= \left\| \sum_{j \in i^{(\ell+1)*}} \mathcal{F}^{-1}(\gamma_i \varphi_j f) \right\|_{L^p} \\ &\leq C_1 \sum_{j \in i^{(\ell+1)*}} \|\mathcal{F}^{-1}(\gamma_i \varphi_j f)\|_{L^p} \end{aligned}$$

for all $i \in I$. There are now two cases. For $p \in [1, \infty]$, Young's inequality $L^1 * L^p \hookrightarrow L^p$ yields

$$\begin{aligned} \|\mathcal{F}^{-1}(\gamma_i \varphi_j f)\|_{L^p} &\leq \|\mathcal{F}^{-1} \gamma_i\|_{L^1} \cdot \|\mathcal{F}^{-1}(\varphi_j f)\|_{L^p} \\ &\leq C_{\mathcal{Q},\Gamma,p} \cdot \|\mathcal{F}^{-1}(\varphi_j f)\|_{L^p}. \end{aligned}$$

For $p \in (0, 1)$, we invoke Corollary 3.14, which yields a constant $C_2 = C_2(d, \ell, p, \mathcal{Q}) > 0$ with

$$\begin{aligned} \|\mathcal{F}^{-1}(\gamma_i \varphi_j f)\|_{L^p} &\leq C_2 \cdot |\det T_i|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p} \|\mathcal{F}^{-1}(\varphi_j f)\|_{L^p} \\ &\leq C_2 C_{\mathcal{Q},\Gamma,p} \cdot \|\mathcal{F}^{-1}(\varphi_j f)\|_{L^p}, \end{aligned}$$

where $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ is semi-structured by definition of L^p -decomposition coverings for $p \in (0, 1)$, cf. Definition 3.6. In the estimate above, we also used the inclusions $\text{supp } \gamma_i \subset \overline{Q_i^{\ell*}} \subset \overline{Q_i^{(\ell+1)*}}$ and $\text{supp } \varphi_j \subset \overline{Q_j} \subset \overline{Q_i^{(\ell+1)*}}$ for all $j \in i^{(\ell+1)*}$. Thus, if we set $C_2 := 1$ for $p \in [1, \infty]$, the estimate above is valid for all $i \in I$ and $p \in (0, \infty]$.

With the “higher order clustering map” $\Theta_{\ell+1}$ as in Lemma 3.9, we finally get, using the solidity of Y ,

$$\begin{aligned} \left\| (\|\mathcal{F}^{-1}(\gamma_j f)\|_{L^p})_{i \in I^{(r)}} \right\|_{Y|_{I^{(r)}}} &\leq C_1 C_2 C_{\mathcal{Q},\Gamma,p} \cdot \left\| \left(\sum_{j \in i^{(\ell+1)*}} \|\mathcal{F}^{-1}(\varphi_j f)\|_{L^p} \right)_{i \in I} \right\|_Y \\ &= C_1 C_2 C_{\mathcal{Q},\Gamma,p} \cdot \left\| \Theta_{\ell+1} \left[(\|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p})_{i \in I} \right] \right\|_Y \\ &\leq C_1 C_2 C_{\mathcal{Q},\Gamma,p} \cdot \|\Theta_{\ell+1}\|_{Y \rightarrow Y} \cdot \left\| (\|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p})_{i \in I} \right\|_Y \\ &= C_1 C_2 C_{\mathcal{Q},\Gamma,p} \cdot \|\Theta_{\ell+1}\|_{Y \rightarrow Y} \cdot \|f\|_{\mathcal{D}_{\mathcal{F},\Phi}(\mathcal{Q},L^p,Y)}. \end{aligned}$$

Summing over $r \in \underline{r_0}$ completes the proof of the first estimate, since $\|\Theta_{\ell+1}\|_{Y \rightarrow Y} \leq \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}^{\ell+1}$ by Lemma 3.9.

Now, let $f \in \mathcal{D}'(\mathcal{O})$ with $\|f\|_{\Gamma, (I^{(r)})_{r \in \underline{r_0}}, L^p, Y} < \infty$ and assume that Γ is an L^p -bounded control system for \mathcal{Q} , i.e. that $\gamma_i \equiv 1$ on Q_i for all $i \in I$. This implies $\varphi_i = \varphi_i \gamma_i$. Thus, in case of $p \in [1, \infty]$, Young's inequality yields

$$\begin{aligned} \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p} &= \|\mathcal{F}^{-1}(\varphi_i \gamma_i f)\|_{L^p} \\ &\leq \|\mathcal{F}^{-1} \varphi_i\|_{L^1} \cdot \|\mathcal{F}^{-1}(\gamma_i f)\|_{L^p} \\ &\leq C_2 C_{\mathcal{Q},\Phi,p} \cdot \|\mathcal{F}^{-1}(\gamma_i f)\|_{L^p} \end{aligned}$$

with $C_2 := 1$. In case of $p \in (0, 1)$, we can use the same constant $C_2 = C_2(d, \ell, p, \mathcal{Q}) > 0$ provided by Corollary 3.14 as above to conclude

$$\begin{aligned} \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p} &= \|\mathcal{F}^{-1}(\varphi_i \gamma_i f)\|_{L^p} \\ &\leq C_2 |\det T_i|^{\frac{1}{p}-1} \|\mathcal{F}^{-1} \varphi_i\|_{L^p} \|\mathcal{F}^{-1}(\gamma_i f)\|_{L^p} \\ &\leq C_2 C_{\mathcal{Q},\Phi,p} \cdot \|\mathcal{F}^{-1}(\gamma_i f)\|_{L^p}, \end{aligned}$$

so that this estimate holds for all $i \in I$ and $p \in (0, \infty]$.

Let $c_i := \|\mathcal{F}^{-1}(\gamma_i f)\|_{L^p}$ for $i \in I$. Since $I = \bigcup_{r=1}^{r_0} I^{(r)}$ and because of $c_i \geq 0$ for all $i \in I$, we have

$$0 \leq c_i \leq \sum_{r=1}^{r_0} (c \cdot \mathbf{1}_{I^{(r)}})_i \text{ for all } i \in I.$$

Since Y is a solid quasi-normed sequence space, there is thus some constant $C_3 = C_3(C_0, r_0) > 0$ (recall that C_0 is a triangle constant for Y) with

$$\begin{aligned} \|f\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)} &= \left\| \left(\|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p} \right)_{i \in I} \right\|_Y \\ &\leq C_2 C_{\mathcal{Q}, \Phi, p} \cdot \|c\|_Y \\ &\leq C_2 C_{\mathcal{Q}, \Phi, p} \cdot \left\| \sum_{r=1}^{r_0} c \cdot \mathbf{1}_{I^{(r)}} \right\|_Y \\ &\leq C_2 C_3 C_{\mathcal{Q}, \Phi, p} \cdot \sum_{r=1}^{r_0} \|c \cdot \mathbf{1}_{I^{(r)}}\|_Y \\ &= C_2 C_3 C_{\mathcal{Q}, \Phi, p} \cdot \|f\|_{\Gamma, (I^{(r)})_r, L^p, Y} < \infty. \end{aligned} \quad \square$$

Using the theorem above, well-definedness of $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$, i.e. independence of the chosen L^p -BAPU is an easy consequence:

Corollary 3.18. *Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open and let $p \in (0, \infty]$. Assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an L^p -decomposition covering of \mathcal{O} with L^p -BAPUs $(\varphi_i)_{i \in I}, (\psi_i)_{i \in I}$ and that Y is \mathcal{Q} -regular.*

Then we have

$$\left\| \left(\|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p} \right)_{i \in I} \right\|_Y \asymp \left\| \left(\|\mathcal{F}^{-1}(\psi_i f)\|_{L^p} \right)_{i \in I} \right\|_Y$$

for all $f \in \mathcal{D}'(\mathcal{O})$, where the implied constants are independent of f . Especially, the left-hand side is finite iff the right-hand side is. \blacktriangleleft

Proof. By Remark 3.16, we know that $\Gamma := (\varphi_i^*)_{i \in I}$ yields an L^p -bounded control system for \mathcal{Q} . By Theorem 3.17, this implies

$$\|\cdot\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)} \asymp \|\cdot\|_{\Gamma, (I), L^p, Y} \asymp \|\cdot\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{Q}, L^p, Y)}$$

for $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_i)_{i \in I}$, where the implied constants only depend on $p, d, \mathcal{Q}, \Phi, \Psi$ and Y . \square

Note that for $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ to be defined at all, we always needed to assume that \mathcal{Q} is an L^p -decomposition covering, i.e. that there is *some* L^p -BAPU for \mathcal{Q} . It is thus important to have sufficient criteria for this to hold.

As we noted after the definition of almost structured coverings, the most important reason for their introduction is that they always possess L^p -BAPUs. This was observed in [4, Proposition 1] for the case of *structured* coverings of $\mathcal{O} = \mathbb{R}^d$. As a slight generalization, the following result was shown in [24, Theorem 2.8]:

Theorem 3.19. Let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ be an *almost structured* covering of the open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$.

Then I is countably infinite and there is a family $\Phi = (\varphi_i)_{i \in I}$ in $C_c^\infty(\mathcal{O})$ such that Φ is an L^p -BAPU for \mathcal{Q} simultaneously for each $p \in (0, \infty]$. In particular, every almost structured covering is an L^p -decomposition covering, for arbitrary $p \in (0, \infty]$. \blacktriangleleft

Now, we know that the Fourier-side decomposition spaces $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ —and thus also the space-side spaces $\mathcal{D}(\mathcal{Q}, L^p, Y)$ —are well-defined for arbitrary L^p -decomposition coverings \mathcal{Q} , in particular for almost structured \mathcal{Q} , if Y is \mathcal{Q} -regular. In the next subsection, we close our investigation of the basic properties of decomposition spaces by showing completeness of $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$.

3.4. Completeness of decomposition spaces. We will now show that the space $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ is indeed a Quasi-Banach space. For the proof, we first establish an equivalent condition for completeness of quasi-normed vector spaces. For a normed vector space X , it is well-known (cf. [8, Theorem 5.1]) that X is complete if and only if “absolute convergence” implies convergence. For quasi-normed vector spaces, we have the following replacement:

Lemma 3.20. *Let $(X, \|\cdot\|)$ be a quasi-normed vector space and let $C \geq 1$ be a triangle constant for $\|\cdot\|$. Then the following are true:*

- (1) *If $x_n \rightarrow x$, then $\|x\| \leq C \cdot \liminf_{n \rightarrow \infty} \|x_n\|$.*
- (2) *We have $\|\sum_{i=1}^n x_i\| \leq \sum_{i=1}^n C^i \|x_i\|$ for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$.*
- (3) *X is complete if there is some $M > 1$ with the following property:*

The series $\sum_{n=1}^{\infty} x_n$ converges (in X) for each $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ with $\|x_n\| \leq M^{-n}$ for all $n \in \mathbb{N}$.

- (4) *Conversely, if X is complete and if $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ satisfies $\sum_{n=1}^{\infty} C^n \|x_n\| < \infty$, then $\sum_{n=1}^{\infty} x_n$ converges and*

$$\left\| \sum_{n=1}^{\infty} x_n \right\| \leq C \cdot \sum_{n=1}^{\infty} C^n \|x_n\|. \quad \blacktriangleleft$$

Remark. Quasi-norms are in general *not* continuous, i.e. $\|x_n - x\| \xrightarrow{n \rightarrow \infty} 0$ does in general *not* imply $\|x_n\| \xrightarrow{n \rightarrow \infty} \|x\|$. Thus, the following proof has to avoid using this property. \blacklozenge

Proof. 1. We have $\|x\| \leq C \cdot [\|x - x_n\| + \|x_n\|]$. Taking the $\liminf_{n \rightarrow \infty}$ proves the claim.

2. We prove the claim by induction on $n \in \mathbb{N}$. For $n = 1$, the claim is trivial, since we have $C \geq 1$. For the induction step, note

$$\begin{aligned} \left\| \sum_{i=1}^{n+1} x_i \right\| &= \left\| x_1 + \sum_{i=2}^{n+1} x_i \right\| \leq C \left[\|x_1\| + \left\| \sum_{i=2}^{n+1} x_i \right\| \right] \\ &\stackrel{(*)}{\leq} C \left[\|x_1\| + \sum_{i=2}^{n+1} C^{i-1} \|x_i\| \right] \\ &= \sum_{i=1}^{n+1} C^i \|x_i\|. \end{aligned}$$

At $(*)$, we implicitly used that the induction hypothesis yields

$$\left\| \sum_{i=2}^{n+1} x_i \right\| = \left\| \sum_{\ell=1}^n x_{\ell+1} \right\| \leq \sum_{\ell=1}^n C^{\ell} \|x_{\ell+1}\| = \sum_{i=2}^{n+1} C^{i-1} \|x_i\|. \quad (3.13)$$

In the remainder of the proof, we will use similar manipulations without further comment.

3. Assume that X satisfies the stated property and let $(y_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in X . Using a trivial induction, we can choose a strictly increasing sequence $(N_n)_{n \in \mathbb{N}}$ satisfying

$$\forall m, \ell \geq N_n : \quad \|y_m - y_{\ell}\| \leq M^{-n}.$$

Define $x_n := y_{N_{n+1}} - y_{N_n}$. Because of $N_{n+1} > N_n$, this yields $\|x_n\| \leq M^{-n}$ for all $n \in \mathbb{N}$, so that the assumption shows that

$$\begin{aligned} x &:= \sum_{n=1}^{\infty} x_n = \lim_{K \rightarrow \infty} \sum_{n=1}^K x_n \\ &= \lim_{K \rightarrow \infty} \sum_{n=1}^K (y_{N_{n+1}} - y_{N_n}) \\ &= \lim_{K \rightarrow \infty} (y_{N_{K+1}} - y_{N_1}) \end{aligned}$$

exists in X . We conclude $y_{N_{K+1}} \rightarrow z := x + y_{N_1}$ as $K \rightarrow \infty$. But this yields

$$\|y_n - z\| \leq C \cdot [\|y_n - y_{N_{n+1}}\| + \|y_{N_{n+1}} - z\|] \xrightarrow{n \rightarrow \infty} 0,$$

so that X is complete.

4. Set $y_n := \sum_{i=1}^n x_i$ for $n \in \mathbb{N}$. Using the second part of the lemma, we get for $N \geq M \geq M_0$ that

$$\begin{aligned} \|y_N - y_M\| &= \left\| \sum_{i=M+1}^N x_i \right\| \\ &\leq \sum_{i=M+1}^N C^{i-M} \|x_i\| \\ &\leq C^{-M} \cdot \sum_{i=M+1}^{\infty} C^i \|x_i\| \\ &\leq \sum_{i=M_0+1}^{\infty} C^i \|x_i\| \xrightarrow{M_0 \rightarrow \infty} 0, \end{aligned}$$

so that $(y_n)_{n \in \mathbb{N}}$ is Cauchy and thus convergent to some $y \in X$ by completeness. But this means $y = \sum_{n=1}^{\infty} x_n$.

Finally, the first and second part of the lemma yield

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} x_n \right\| &\leq C \cdot \liminf_{N \rightarrow \infty} \left\| \sum_{n=1}^N x_n \right\| \\ &\leq C \cdot \liminf_{N \rightarrow \infty} \sum_{n=1}^N C^n \|x_n\| \\ &= C \cdot \sum_{n=1}^{\infty} C^n \|x_n\|. \end{aligned} \quad \square$$

Now, we can show that $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ is a Quasi-Banach space which embeds continuously into $\mathcal{D}'(\mathcal{O})$.

Theorem 3.21. Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open and let $p \in (0, \infty]$. Assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an L^p -decomposition covering of \mathcal{O} and that $Y \leq \mathbb{C}^I$ is \mathcal{Q} -regular.

Then $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ is a Quasi-Banach space which embeds continuously into $\mathcal{D}'(\mathcal{O})$. The triangle constant C for $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ satisfies $C \leq C^{(p)} C_Y$, where C_Y is a triangle constant for Y and $C^{(p)}$ is a triangle constant for L^p .

If Y is a Banach space (instead of a Quasi-Banach space) and if $p \in [1, \infty]$, then $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ is a Banach space, i.e. $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)}$ is a norm. \blacktriangleleft

Proof. It is clear that $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ is closed under multiplication with complex scalars and that $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)}$ is homogeneous.

Let $(\varphi_i)_{i \in I}$ be an L^p -BAPU for \mathcal{Q} which is used to define $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)}$. For $f, g \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ and $i \in I$, we have

$$\mathcal{F}^{-1}[\varphi_i(f + g)] = \mathcal{F}^{-1}(\varphi_i f) + \mathcal{F}^{-1}(\varphi_i g) \in L^p(\mathbb{R}^d)$$

with

$$\|\mathcal{F}^{-1}[\varphi_i(f + g)]\|_{L^p} \leq C^{(p)} \cdot [\|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p} + \|\mathcal{F}^{-1}(\varphi_i g)\|_{L^p}].$$

By solidity of Y , this implies $(\|\mathcal{F}^{-1}[\varphi_i(f+g)]\|_{L^p})_{i \in I} \in Y$ with

$$\begin{aligned} \|f+g\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} &= \left\| (\|\mathcal{F}^{-1}[\varphi_i(f+g)]\|_{L^p})_{i \in I} \right\|_Y \\ &\leq C^{(p)} \cdot \left\| (\|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p} + \|\mathcal{F}^{-1}(\varphi_i g)\|_{L^p})_{i \in I} \right\|_Y \\ &\leq C^{(p)} C_Y \cdot \left[\left\| (\|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p})_{i \in I} \right\|_Y + \left\| (\|\mathcal{F}^{-1}(\varphi_i g)\|_{L^p})_{i \in I} \right\|_Y \right] \\ &= C^{(p)} C_Y \cdot [\|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} + \|g\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)}] < \infty. \end{aligned}$$

This shows that $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ is a vector space and that $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)}$ is a quasi-norm (once we know that $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)}$ is positive definite, which we show below). If Y is a Banach space and if $p \in [1, \infty]$, we can take $C^{(p)} = C_Y = 1$, so that $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)}$ is a genuine norm.

Now, let us prove $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y) \hookrightarrow \mathcal{D}'(\mathcal{O})$. To this end, let $K \subset \mathcal{O}$ be an arbitrary compact set. Lemma 2.4 shows that $(Q_i^\circ)_{i \in I}$ covers \mathcal{O} . Since $K \subset \mathcal{O}$ is compact, there are finitely many $i_1, \dots, i_n \in I$ with $K \subset \bigcup_{\ell=1}^n Q_{i_\ell}^\circ \subset \bigcup_{\ell=1}^n Q_{i_\ell}$. The set $I_K := \{i_1, \dots, i_n\}^* \subset I$ is finite and Lemma 2.4 implies $\varphi_{I_K} \equiv 1$ on K .

Now, choose $u = (u_i)_{i \in I}$ as in Lemma 3.8, so that $Y \hookrightarrow \ell_u^\infty(I)$. Let us set $C_K := \min_{i \in I_K} u_i > 0$. For arbitrary $\varphi \in C_c^\infty(\mathcal{O})$ with $\text{supp } \varphi \subset K$ and $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y) \subset \mathcal{D}'(\mathcal{O})$, we now have

$$\begin{aligned} |\langle f, \varphi \rangle| &= \left| \langle f, \sum_{i \in I_K} \varphi_i \varphi \rangle \right| = \left| \sum_{i \in I_K} \langle \varphi_i f, \varphi \rangle \right| \\ &\leq \sum_{i \in I_K} |\langle \mathcal{F}^{-1}(\varphi_i f), \widehat{\varphi} \rangle|. \end{aligned}$$

There are now two cases: For $p \in [1, \infty]$, we can apply Hölder's inequality to conclude

$$|\langle \mathcal{F}^{-1}(\varphi_i f), \widehat{\varphi} \rangle| \leq \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p} \cdot \|\widehat{\varphi}\|_{L^{p'}}.$$

In case of $p \in (0, 1)$, we observe $\text{supp } \widehat{\mathcal{F}^{-1}(\varphi_i f)} = \text{supp } (\varphi_i f) \subset \overline{Q_i}$ and apply Corollary 3.3 with $q = \infty$ to conclude

$$\|\mathcal{F}^{-1}(\varphi_i f)\|_{L^\infty} \leq [\lambda(\overline{Q_i})]^{\frac{1}{p}} \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p}$$

and hence

$$\begin{aligned} |\langle \mathcal{F}^{-1}(\varphi_i f), \widehat{\varphi} \rangle| &\leq [\lambda(\overline{Q_i})]^{\frac{1}{p}} \cdot \|\widehat{\varphi}\|_{L^1} \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p} \\ &\leq C'_K \cdot \|\widehat{\varphi}\|_{L^1} \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p} \end{aligned}$$

for all $i \in I_K$, with $C'_K := \max_{i \in I_K} [\lambda(\overline{Q_i})]^{1/p}$.

In any of the two cases, there is thus an exponent $r \in [1, \infty]$ and some constant $C'_K > 0$ satisfying

$$\begin{aligned} |\langle f, \varphi \rangle| &\leq C'_K \cdot \|\widehat{\varphi}\|_{L^r} \cdot \sum_{i \in I_K} \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p} \\ &\leq \frac{C'_K}{C_K} \cdot \|\widehat{\varphi}\|_{L^r} \cdot \sum_{i \in I_K} [u_i \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p}] \\ &\leq \frac{C'_K}{C_K} \cdot \|\widehat{\varphi}\|_{L^r} \cdot \left\| (\|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p})_{i \in I} \right\|_{\ell_u^\infty} \cdot |I_K| \\ &\leq \frac{C'_K \cdot |I_K|}{C_K} \cdot \|\widehat{\varphi}\|_{L^r} \cdot \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)}. \end{aligned} \tag{3.14}$$

Here, the term $\|\widehat{\varphi}\|_{L^r}$ is finite because of $\varphi \in C_c^\infty(\mathcal{O}) \leq \mathcal{S}(\mathbb{R}^d)$. We recall from above the assumption $\text{supp } \varphi \subset K$, where $K \subset \mathcal{O}$ was an *arbitrary* compact subset of \mathcal{O} . In particular, the above estimate proves $f \equiv 0$ as an element of $\mathcal{D}'(\mathcal{O})$ if $\|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} = 0$. Hence, $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)}$ is positive definite.

Now, if $(f_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ converging in $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ to $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$, then the above estimate easily implies $|\langle f_n - f, \varphi \rangle| \xrightarrow{n \rightarrow \infty} 0$ for all $\varphi \in C_c^\infty(\mathcal{O})$ and hence $f_n \rightarrow f$ with convergence in $\mathcal{D}'(\mathcal{O})$. This establishes the continuous embedding $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y) \hookrightarrow \mathcal{D}'(\mathcal{O})$. We note

that it suffices to consider sequences to prove this continuity, since the topology of the quasi-normed vector space $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ is first countable.

It remains to prove completeness of $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$. To this end, note that Lemma 3.20 shows that it suffices to consider a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ with $\|f_n\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} \leq (2C_Y C^{(p)})^{-n}$ for all $n \in \mathbb{N}$ and to show that the sequence $g_N := \sum_{n=1}^N f_n$ converges to some $g \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$.

Let us set

$$\theta_i^{(n)} := \|\mathcal{F}^{-1}(\varphi_i f_n)\|_{L^p}$$

for $i \in I$ and $n \in \mathbb{N}$ and note $\theta^{(n)} := (\theta_i^{(n)})_{i \in I} \in Y$ with $\|\theta^{(n)}\|_Y = \|f_n\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} \leq (2C_Y C^{(p)})^{-n}$. Since Y is complete and because of

$$\sum_{n=N}^{\infty} C_Y^{n-N+1} \left\| \left(C^{(p)} \right)^n \theta^{(n)} \right\|_Y \leq \sum_{n=N}^{\infty} \left(C_Y C^{(p)} \right)^n \left\| \theta^{(n)} \right\|_Y \leq \sum_{n=N}^{\infty} 2^{-n} < \infty,$$

Lemma 3.20 implies $\sum_{n=N}^{\infty} (C^{(p)})^n \theta^{(n)} \in Y$ for all $N \in \mathbb{N}$, and also

$$\left\| \sum_{n=N}^{\infty} \left(C^{(p)} \right)^n \theta^{(n)} \right\|_Y \leq C_Y \cdot \sum_{n=N}^{\infty} C_Y^{n-N+1} \left\| \left(C^{(p)} \right)^n \theta^{(n)} \right\|_Y \leq C_Y^{2-N} \sum_{n=N}^{\infty} 2^{-n} \xrightarrow{N \rightarrow \infty} 0. \quad (3.15)$$

Note that implicitly an argument as in equation (3.13) was used here. In particular, since $Y \hookrightarrow \ell_u^\infty(I)$ for a suitable weight u (cf. Lemma 3.8), we get $\sum_{n=1}^{\infty} (C^{(p)})^n \theta_i^{(n)} < \infty$ for all $i \in I$.

Now, we note that equation (3.14) implies for each compact $K \subset \mathcal{O}$ and each $\varphi \in C_c^\infty(\mathcal{O})$ with $\text{supp } \varphi \subset K$ the estimate

$$\sum_{n=1}^{\infty} |\langle f_n, \varphi \rangle| \leq \frac{C'_K \cdot |I_K|}{C_K} \cdot \|\widehat{\varphi}\|_{L^r} \cdot \sum_{n=1}^{\infty} \|f_n\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} < \infty.$$

Thus, $g(\varphi) := \sum_{n=1}^{\infty} \langle f_n, \varphi \rangle$ converges for every $\varphi \in C_c^\infty(\mathcal{O})$. By [16, Theorem 6.17], this implies $g \in \mathcal{D}'(\mathcal{O})$. It remains to show $g \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ and $\|g_N - g\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} \rightarrow 0$.

We recall from [16, Theorem 7.23] that the inverse Fourier transform $\mathcal{F}^{-1}(\varphi_i [g - g_N])$ is given by

$$\begin{aligned} [\mathcal{F}^{-1}(\varphi_i [g - g_N])](x) &= \langle \varphi_i [g - g_N], e_x \rangle = \langle g - g_N, \varphi_i e_x \rangle \\ &= \sum_{n=N+1}^{\infty} \langle f_n, \varphi_i e_x \rangle = \sum_{n=N+1}^{\infty} \langle \varphi_i f_n, e_x \rangle \\ &= \sum_{n=1}^{\infty} [\mathcal{F}^{-1}(\varphi_i f_{n+N})](x) \end{aligned} \quad (3.16)$$

for all $x \in \mathbb{R}^d$, with $e_z(y) = e^{2\pi i \langle z, y \rangle}$ for $y, z \in \mathbb{R}^d$.

But above, we saw $\sum_{n=1}^{\infty} (C^{(p)})^n \|\mathcal{F}^{-1}(\varphi_i f_n)\|_{L^p} = \sum_{n=1}^{\infty} (C^{(p)})^n \theta_i^{(n)} < \infty$ for all $i \in I$, which in particular implies

$$\sum_{n=1}^{\infty} (C^{(p)})^n \|\mathcal{F}^{-1}(\varphi_i f_{n+N})\|_{L^p} = (C^{(p)})^{-N} \sum_{\ell=N+1}^{\infty} (C^{(p)})^{\ell} \|\mathcal{F}^{-1}(\varphi_i f_{\ell})\|_{L^p} < \infty.$$

By Lemma 3.20, this yields $\sum_{n=1}^{\infty} \mathcal{F}^{-1}(\varphi_i f_{n+N}) \in L^p(\mathbb{R}^d)$, since $C^{(p)}$ is a triangle constant for L^p . Furthermore, the same lemma shows

$$\begin{aligned} 0 \leq \gamma_i^{(N)} &:= \left\| \sum_{n=1}^{\infty} \mathcal{F}^{-1}(\varphi_i f_{n+N}) \right\|_{L^p} \leq C^{(p)} \sum_{n=1}^{\infty} \left(C^{(p)} \right)^n \left\| \mathcal{F}^{-1}(\varphi_i f_{n+N}) \right\|_{L^p} \\ &= \sum_{n=N+1}^{\infty} \left(C^{(p)} \right)^{n-N+1} \left\| \mathcal{F}^{-1}(\varphi_i f_n) \right\|_{L^p} \\ &\leq \sum_{n=N+1}^{\infty} \left(C^{(p)} \right)^n \left\| \mathcal{F}^{-1}(\varphi_i f_n) \right\|_{L^p} \\ &= \sum_{n=N+1}^{\infty} \left(C^{(p)} \right)^n \theta_i^{(n)}. \end{aligned}$$

Note that convergence in $L^p(\mathbb{R}^d)$ implies convergence almost everywhere for a subsequence. Together with equation (3.16), this implies $\mathcal{F}^{-1}(\varphi_i [g - g_N]) = \sum_{n=1}^{\infty} \mathcal{F}^{-1}(\varphi_i f_{n+N}) \in L^p(\mathbb{R}^d)$ and thus also $\gamma_i^{(N)} = \left\| \mathcal{F}^{-1}(\varphi_i [g - g_N]) \right\|_{L^p}$.

But above, we saw $\sum_{n=N+1}^{\infty} \left(C^{(p)} \right)^n \theta_i^{(n)} \in Y$. By solidity of Y and using equation (3.15), we conclude $\gamma^{(N)} := (\gamma_i^{(N)})_{i \in I} \in Y$ with

$$\|g - g_N\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} = \left\| \gamma^{(N)} \right\|_Y \leq \left\| \sum_{n=N+1}^{\infty} \left(C^{(p)} \right)^n \theta^{(n)} \right\|_Y \xrightarrow[N \rightarrow \infty]{\text{eq. (3.15)}} 0.$$

In particular, we get $g = (g - g_N) + g_N \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ and the above estimate proves $g = \lim_{N \rightarrow \infty} g_N$ with convergence in $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$. As noted above, this completes the proof. \square

We end this section with an example which shows that the completeness proved above is not trivial. More precisely, we show that completeness can fail if the reservoir $\mathcal{D}'(\mathcal{O})$ is replaced by $\mathcal{S}'(\mathbb{R}^d)$.

Example 3.22. In the following, we provide a specific example showing that the space $\mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^p, Y)$ as defined in Remark 3.13 is in general *not* complete. Specifically, we will take $p = 1$ and $Y = \ell_u^1$. The covering which we will use is the “uniform covering” which is usually used to define modulation spaces; but in our case, the weight u decays (for $x \rightarrow \infty$) much faster than the weights used to define the classical modulation spaces.

Let $I := \mathbb{Z}$, $T_i := \text{id}_{\mathbb{R}}$ and $b_i := i$ for $i \in \mathbb{Z}$. Furthermore, define $Q := (-\frac{3}{4}, \frac{3}{4})$ and $P := (-\frac{5}{8}, \frac{5}{8})$, as well as

$$Q_i := T_i Q + b_i = \left(i - \frac{3}{4}, i + \frac{3}{4} \right).$$

It is then easy to see $\bigcup_{i \in I} (T_i P + b_i) = \mathbb{R}$ and that $x \in Q_i \cap Q_j \neq \emptyset$ implies

$$i - \frac{3}{4} < x < j + \frac{3}{4}$$

and hence $i - j < \frac{6}{4} < 2$. Since $i - j \in \mathbb{Z}$, we conclude $i - j \leq 1$. By symmetry we get $|i - j| \leq 1$ and thus $i^* \subset \{i - 1, i, i + 1\}$. This shows that $\mathcal{Q} = (Q_i)_{i \in I}$ is a structured admissible covering of \mathbb{R} .

Now consider the weight $u_i := 10^{-i}$ for $i \in \mathbb{Z}$ and note that because of the estimate

$$\frac{u_i}{u_j} = 10^{j-i} \leq 10^{|j-i|} \leq 10,$$

which is valid for all $i \in I$ and $j \in i^* \subset \{i - 1, i, i + 1\}$, the weight u is \mathcal{Q} -moderate.

Theorem 3.19 guarantees the existence of an L^p -BAPU $(\varphi_i)_{i \in I}$ for \mathcal{Q} . Note that for $i \in I$ we have

$$\begin{aligned} \bigcup_{j \in I \setminus \{i\}} Q_j &\subset \left(-\infty, (i-1) + \frac{3}{4} \right) \cup \left((i+1) - \frac{3}{4}, \infty \right) \\ &= \left(-\infty, i - \frac{1}{4} \right) \cup \left(i + \frac{1}{4}, \infty \right). \end{aligned}$$

Together with $\varphi_j(\xi) = 0$ for $\xi \in \mathbb{R} \setminus Q_j$ and $\sum_{i \in I} \varphi_i(\xi) = 1$ for all $\xi \in \mathbb{R}$, this implies $\varphi_i(\xi) = 1$ for $\xi \in [i - \frac{1}{4}, i + \frac{1}{4}]$ for all $i \in I = \mathbb{Z}$.

Now choose a nonnegative function $\psi \in C_c^\infty((-\frac{1}{4}, \frac{1}{4})) \setminus \{0\}$ and define $f_n := \sum_{j=1}^n 4^j \cdot L_j \psi$ for $n \in \mathbb{N}$. Because of

$$\text{supp}(L_n \psi) \subset \left(n - \frac{1}{4}, n + \frac{1}{4}\right) \subset \left(\bigcup_{j \in I \setminus \{n\}} Q_j\right)^c$$

it is easy to see that

$$\varphi_i \cdot L_n \psi = \begin{cases} 0, & \text{if } i \neq n, \\ L_i \psi, & \text{if } i = n \end{cases}$$

holds for $i, n \in \mathbb{Z}$. For $n \geq m \geq m_0$ we thus get

$$\begin{aligned} \|f_n - f_m\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^1, \ell_u^1)} &= \sum_{i \in \mathbb{Z}} 10^{-i} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \sum_{j=m+1}^n 4^j \cdot L_j \psi \right) \right\|_{L^1} \\ &= \sum_{i=m+1}^n 10^{-i} 4^i \cdot \|\mathcal{F}^{-1}(L_i \psi)\|_{L^1} \\ &\leq \|\mathcal{F}^{-1} \psi\|_{L^1} \cdot \sum_{i=m_0+1}^{\infty} \left(\frac{4}{10}\right)^i \xrightarrow{m_0 \rightarrow \infty} 0, \end{aligned}$$

so that $(\mathcal{F}^{-1} f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^1, \ell_u^1)$, since $\mathcal{F} : \mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^1, \ell_u^1) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^1, \ell_u^1)$ is isometric (but not necessarily surjective).

Let us assume that there is some $f \in \mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^1, \ell_u^1) \subset \mathcal{S}'(\mathbb{R})$ with $\mathcal{F}^{-1} f_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^1, \ell_u^1)} f$. Using the continuity of

$$\mathcal{F} : \mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^1, \ell_u^1) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^1, \ell_u^1)$$

and the continuous embedding $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^1, \ell_u^1) \hookrightarrow \mathcal{D}'(\mathbb{R})$ (cf. Theorem 3.21), we conclude

$$\langle \widehat{f}, g \rangle_{\mathcal{D}', \mathcal{D}} = \lim_{n \rightarrow \infty} \langle f_n, g \rangle_{\mathcal{D}', \mathcal{D}} \quad \text{for all } g \in C_c^\infty(\mathbb{R}).$$

Furthermore, by definition of the topology on $\mathcal{S}(\mathbb{R})$, by [8, Proposition 5.15] and because of $\widehat{f} \in \mathcal{S}'(\mathbb{R})$ (since $f \in \mathcal{S}'(\mathbb{R})$), there exists some $N \in \mathbb{N}$ and some $C > 0$ such that

$$\left| \langle \widehat{f}, g \rangle_{\mathcal{S}', \mathcal{S}} \right| \leq C \cdot \sup_{\substack{\alpha \in \mathbb{N}_0 \\ \alpha \leq N}} \sup_{\xi \in \mathbb{R}} (1 + |\xi|)^N \cdot |(\partial^\alpha g)(\xi)|$$

holds for all $g \in \mathcal{S}(\mathbb{R})$.

For $n \in \mathbb{N}$ and $g_n := L_n \psi$ we have

$$\text{supp } g_n \subset \left(n - \frac{1}{4}, n + \frac{1}{4}\right) \subset [0, n + 1]$$

and thus

$$\begin{aligned} \sup_{\substack{\alpha \in \mathbb{N}_0 \\ \alpha \leq N}} \sup_{\xi \in \mathbb{R}} (1 + |\xi|)^N \cdot |(\partial^\alpha g_n)(x)| &\leq (n + 2)^N \cdot \sup_{\substack{\alpha \in \mathbb{N}_0 \\ \alpha \leq N}} \sup_{\xi \in \mathbb{R}} |(\partial^\alpha \psi)(\xi)| \\ &= (n + 2)^N \cdot C_{\psi, N} \end{aligned}$$

for some constant $C_{\psi, N} \in (0, \infty)$.

But because of $\text{supp}(L_n \psi) \cap \text{supp}(L_i \psi) = \emptyset$ for $i, n \in \mathbb{Z}$ with $i \neq n$ we have, for $m \geq n$, the identity

$$\langle f_m, g_n \rangle_{\mathcal{S}', \mathcal{S}} = \sum_{j=1}^m 4^j \langle L_j \psi, L_n \psi \rangle_{\mathcal{S}', \mathcal{S}} = 4^n \cdot \langle \psi, \psi \rangle_{\mathcal{S}', \mathcal{S}}$$

and thus

$$\begin{aligned} 4^n \cdot \|\psi\|_{L^2}^2 &= \lim_{m \rightarrow \infty} \left| \langle f_m, g_n \rangle_{\mathcal{S}', \mathcal{S}} \right| = \left| \langle \widehat{f}, g_n \rangle_{\mathcal{S}', \mathcal{S}} \right| \\ &\leq C \cdot \sup_{\substack{\alpha \in \mathbb{N}_0 \\ \alpha \leq N}} \sup_{\xi \in \mathbb{R}} (1 + |\xi|)^N \cdot |(\partial^\alpha g_n)(\xi)| \\ &\leq CC_{\psi, N} \cdot (n + 2)^N \end{aligned}$$

for all $n \in \mathbb{N}$, a contradiction.

Thus, there is no $f \in \mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^1, \ell_u^1)$ with $\|f - \mathcal{F}^{-1}f_n\|_{\mathcal{D}(\mathcal{Q}, L^1, \ell_u^1)} \xrightarrow{n \rightarrow \infty} 0$, so that $\mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^1, \ell_u^1)$ is *not* complete. \blacklozenge

4. NESTED SEQUENCE SPACES

Recall from Section 1.3 that most of our embedding results will consist in showing that an embedding between certain sequence spaces is sufficient (or necessary) for the existence of an embedding

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z).$$

Furthermore, as seen e.g. in Theorems 1.1 and 1.3, these sequence spaces are often of a “nested” nature, so that the (quasi)-norm is for example given by

$$\|(x_i)_{i \in I}\|_{X([\ell_u^q(I_k)]_{k \in K})} = \left\| \left(\|(u_{k,i} \cdot x_i)_{i \in I_k}\|_{\ell^q} \right)_{k \in K} \right\|_X.$$

Thus, the present section is devoted to studying spaces of this type. In particular, we will consider the existence of embeddings between such sequence spaces.

We begin with a rather general definition:

Definition 4.1. Let K be an index set and let $X \leq \mathbb{C}^K$ be a solid sequence space. Assume that for each $k \in K$, we are given a set I_k and a solid sequence space $X_k \leq \mathbb{C}^{I_k}$. Then, setting $I := \bigcup_{k \in K} I_k$, we define the **nested sequence space**

$$X([X_k]_{k \in K}) := \left\{ x = (x_i)_{i \in I} \in \mathbb{C}^I \mid (\forall k \in K : (x_i)_{i \in I_k} \in X_k) \text{ and } \left(\|(x_i)_{i \in I_k}\|_{X_k} \right)_{k \in K} \in X \right\}.$$

If the context makes the intended meaning clear, we will also write $X(X_k) := X([X_k]_{k \in K})$.

Finally, for $x = (x_i)_{i \in I} \in X([X_k]_{k \in K})$, we define

$$\|x\|_{X([X_k]_{k \in K})} := \left\| \left[\|(x_i)_{i \in I_k}\|_{X_k} \right]_{k \in K} \right\|_X. \quad \blacktriangleleft$$

Remark. We will often have $X_k = \ell_{w_k}^{q_k}(I_k)$ for a suitable weight $w_k = (w_{k,i})_{i \in I_k}$. In this case, we will often write $X([\ell_w^{q_k}(I_k)]_{k \in K})$ instead of the (more correct) $X([\ell_{w_k}^{q_k}(I_k)]_{k \in K})$. \blacklozenge

The first question is of course whether $X([X_k]_{k \in K})$ is always a solid sequence space over the index set $I = \bigcup_{k \in K} I_k$, or even whether it is a vector space. Under very weak assumptions, this is indeed the case:

Lemma 4.2. *In the setting of Definition 4.1, let $C_k \geq 1$ be a triangle constant for X_k for each $k \in K$. If*

$$C := \sup_{k \in K} C_k$$

is finite, then $X([X_k]_{k \in K})$ is a solid sequence space over I , with triangle constant $C \cdot C_0$, where $C_0 \geq 1$ is a triangle constant for X . \blacktriangleleft

Remark. (1) Under the assumptions from above, it is even true that $X([X_k]_{k \in K})$ is complete if X and each X_k are complete. Since we will not need this fact, we omit the proof.

(2) If $X_k = \ell_{u^{(k)}}^{q_k}(I_k)$ for a certain weight $u^{(k)} = (u_i^{(k)})_{i \in I_k}$, then [10, Exercise 1.1.5(d)] implies that a triangle constant for X_k is given by $C_k = \max \left\{ 1, 2^{q_k^{-1}-1} \right\}$. Thus, if $\inf_{k \in K} q_k > 0$, then the lemma is applicable and shows that $X([\ell_{u^{(k)}}^{q_k}]_{k \in K})$ is a solid sequence space over I . \blacklozenge

Proof. For brevity, write $Y := X([X_k]_{k \in K})$. It is clear that Y is closed under multiplication with complex scalars and that $\|\cdot\|_Y$ is homogeneous.

Furthermore, $\|\cdot\|_Y$ is definite: Assume that $x = (x_i)_{i \in I} \in Y$ satisfies $\|x\|_Y = 0$. For arbitrary $i_0 \in I$, there is $k_0 \in K$ with $i_0 \in I_{k_0}$. But $\|x\|_Y = 0$ implies $\|(x_i)_{i \in I_{k_0}}\|_{X_{k_0}} = 0$ and thus $x_{i_0} = 0$. Since $i_0 \in I$ was arbitrary, we see $x = 0$.

To see that Y is solid, let $x = (x_i)_{i \in I} \in \mathbb{C}^I$ and $y = (y_i)_{i \in I} \in Y$ with $|x_i| \leq |y_i|$ for all $i \in I$. Since each X_k is solid with $(y_i)_{i \in I_k} \in X_k$, we get $(x_i)_{i \in I_k} \in X_k$ and $\varrho_k := \|(x_i)_{i \in I_k}\|_{X_k} \leq \|(y_i)_{i \in I_k}\|_{X_k} =: \theta_k$. But $y \in Y$ means $\theta = (\theta_k)_{k \in K} \in X$. By solidity of X , we get $\varrho = (\varrho_k)_{k \in K} \in X$ and hence $x \in Y$, with $\|x\|_Y = \|\varrho\|_X \leq \|\theta\|_X = \|y\|_Y$, as desired.

Finally, to show that Y is a vector space and that $\|\cdot\|_Y$ is a (quasi)-norm, let $x = (x_i)_{i \in I} \in Y$ and $y = (y_i)_{i \in I} \in Y$ and define $x^{(k)} := \|(x_i)_{i \in I_k}\|_{X_k}$ and $y^{(k)} := \|(y_i)_{i \in I_k}\|_{X_k}$ for $k \in K$. By definition of Y , we have $(x^{(k)})_{k \in K}, (y^{(k)})_{k \in K} \in X$ and thus also $(C \cdot [x^{(k)} + y^{(k)}])_{k \in K} \in X$. But we also have

$$z^{(k)} := \|(x_i + y_i)_{i \in I_k}\|_{X_k} \leq C_k \cdot \left[\|(x_i)_{i \in I_k}\|_{X_k} + \|(y_i)_{i \in I_k}\|_{X_k} \right] \leq C \cdot [x^{(k)} + y^{(k)}]$$

for all $k \in K$. By solidity of X , this yields $(z^{(k)})_{k \in K} \in X$ and thus $x + y \in Y$ with

$$\begin{aligned} \|x + y\|_Y &= \|(z^{(k)})_{k \in K}\|_X \\ &\leq \left\| \left(C \cdot [x^{(k)} + y^{(k)}] \right)_{k \in K} \right\|_X \\ &\leq C \cdot C_0 \cdot (\|(x^{(k)})_{k \in K}\|_X + \|(y^{(k)})_{k \in K}\|_X) \\ &= CC_0 \cdot (\|x\|_Y + \|y\|_Y). \end{aligned}$$

Thus, Y is a vector space and CC_0 is a triangle constant for $\|\cdot\|_Y$. \square

An important property of (solid) sequence spaces is the Fatou property, which is an abstraction of the conclusion of Fatou's lemma for the spaces ℓ_u^q to general (solid) sequence spaces:

Definition 4.3. Let $X \leq \mathbb{C}^I$ be a solid sequence space. We say that X has the **Fatou property**, if for each bounded sequence $(x^{(n)})_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ with $x^{(n)} = (x_i^{(n)})_{i \in I}$ and

$$x_i := \liminf_{n \rightarrow \infty} |x_i^{(n)}| \quad \text{for } i \in I,$$

we have $x = (x_i)_{i \in I} \in X$ with $\|x\|_X \leq \liminf_{n \rightarrow \infty} \|x^{(n)}\|_X$. \blacktriangleleft

For later use, we need to know that the Fatou property is inherited by nested sequence spaces and also by weighted sequence spaces:

Definition 4.4. Let $X \leq \mathbb{C}^I$ be a sequence space and let $u = (u_i)_{i \in I} \in (0, \infty)^I$ be a weight. The **weighted sequence space** X_u is given by

$$X_u := \{(x_i)_{i \in I} \in \mathbb{C}^I \mid (u_i \cdot x_i)_{i \in I} \in X\} \quad \text{with} \quad \|(x_i)_{i \in I}\|_{X_u} := \|(u_i \cdot x_i)_{i \in I}\|_X. \quad \blacktriangleleft$$

Remark. It is easy to see that X_u is a solid sequence space if X is, even with the same triangle constant. Furthermore, X is complete iff X_u is. \blacklozenge

Now, we can state and prove inheritability of the Fatou property:

Lemma 4.5. (1) If $X \leq \mathbb{C}^I$ is a solid sequence space with the Fatou property and $u = (u_i)_{i \in I}$ is a weight, then X_u has the Fatou property.

(2) In the setting of Lemma 4.2, if X and each X_k satisfy the Fatou property, then so does $X([X_k]_{k \in K})$. \blacktriangleleft

Proof. 1. Let $(x^{(n)})_{n \in \mathbb{N}} \in X_u^{\mathbb{N}}$ be a bounded sequence with $x^{(n)} = (x_i^{(n)})_{i \in I}$. For

$$y^{(n)} := (y_i^{(n)})_{i \in I} := (u_i \cdot x_i^{(n)})_{i \in I},$$

this means that $(y^{(n)})_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ is a bounded sequence. Thus, for

$$y_i := \liminf_{n \rightarrow \infty} |y_i^{(n)}| = u_i \cdot \liminf_{n \rightarrow \infty} |x_i^{(n)}|,$$

we have $y = (y_i)_{i \in I} \in X$ with

$$\|y\|_X \leq \liminf_{n \rightarrow \infty} \|y^{(n)}\|_X = \liminf_{n \rightarrow \infty} \|x^{(n)}\|_{X_u}.$$

For $x_i := \liminf_{n \rightarrow \infty} |x_i^{(n)}|$, this yields $x = (x_i)_{i \in I} \in X_u$ and $\|x\|_{X_u} = \|y\|_X \leq \liminf_{n \rightarrow \infty} \|x^{(n)}\|_{X_u}$, as desired.

2. For brevity, write $Y := X([X_k]_{k \in K})$. Let $(x^{(n)})_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ be a bounded sequence, with $x^{(n)} = (x_i^{(n)})_{i \in I}$. By solidity of X and because of Lemma 3.8, we have $X \hookrightarrow \ell_u^\infty(I)$ for a certain weight u on I . In particular, we get

$$z_k^{(n)} := \|(x_i^{(n)})_{i \in I_k}\|_{X_k} \lesssim_k \left\| \left[\|(x_i^{(n)})_{i \in I_\ell}\|_{X_\ell} \right]_{\ell \in K} \right\|_X = \|x^{(n)}\|_Y \leq C$$

for each $k \in K$, so that for each $k \in K$, the sequence $(y^{(k,n)})_{n \in \mathbb{N}}$, given by $y^{(k,n)} := (x_i^{(n)})_{i \in I_k}$ is bounded in X_k . Furthermore, we have

$$\|(z_k^{(n)})_{k \in K}\|_X = \|x^{(n)}\|_Y \leq C$$

for all $n \in \mathbb{N}$, so that the sequence $(z^{(n)})_{n \in \mathbb{N}} \in X^{\mathbb{N}}$, given by $z^{(n)} = (z_k^{(n)})_{k \in K}$, is bounded.

Now, for $i \in I$, define $x_i := \liminf_{n \rightarrow \infty} |x_i^{(n)}|$ and for $k \in K$, let $z_k := \liminf_{n \rightarrow \infty} z_k^{(n)}$. Using the Fatou property of X_k , we get $(x_i)_{i \in I_k} \in X_k$, with

$$0 \leq \|(x_i)_{i \in I_k}\|_{X_k} \leq \liminf_{n \rightarrow \infty} \|(x_i^{(n)})_{i \in I_k}\|_{X_k} = \liminf_{n \rightarrow \infty} z_k^{(n)} = z_k.$$

But the Fatou property for X yields $z = (z_k)_{k \in K} \in X$ with

$$\|z\|_X \leq \liminf_{n \rightarrow \infty} \|z^{(n)}\|_X = \liminf_{n \rightarrow \infty} \|x^{(n)}\|_Y.$$

By solidity of X , this finally yields $\left(\|(x_i)_{i \in I_k}\|_{X_k} \right)_{k \in K} \in X$ and thus $x = (x_i)_{i \in I} \in Y$ with

$$\|x\|_Y = \left\| \left[\|(x_i)_{i \in I_k}\|_{X_k} \right]_{k \in K} \right\|_X \leq \|(z_k)_{k \in K}\|_X \leq \liminf_{n \rightarrow \infty} \|x^{(n)}\|_Y.$$

This completes the proof. \square

As noted above, we are interested in embeddings between sequence spaces. Our next lemma shows that if the “target space” of the embedding satisfies the Fatou property, it suffices to verify boundedness of an embedding between *solid* sequence spaces on the space of finitely supported sequences. Although rather technical, this fact will be very helpful for us in the remainder of the paper.

Lemma 4.6. *Let I be a countable(!) set and let $X, Y \leq \mathbb{C}^I$ be solid sequence spaces. If Y satisfies the Fatou property, then $\iota : X \hookrightarrow Y$ is well-defined and bounded if and only if*

$$\iota_0 : X \cap \ell_0(I) \hookrightarrow Y$$

is well-defined and bounded. In this case, $\|\iota\| = \|\iota_0\|$.

*Here, $\ell_0(I) = \langle \delta_i \mid i \in I \rangle$ is the space of **finitely supported sequences** on I .* \blacktriangleleft

Proof. The implication “ \Rightarrow ” and the estimate $\|\iota_0\| \leq \|\iota\|$ are trivial. Hence, we only prove “ \Leftarrow ”. By assumption, $I = \bigcup_{n \in \mathbb{N}} I^{(n)}$ for certain finite subsets $I^{(n)} \subset I$ with $I^{(n)} \subset I^{(n+1)}$ for all $n \in \mathbb{N}$. Let $x = (x_i)_{i \in I} \in X$ be arbitrary and define $x^{(n)} := x \cdot \mathbf{1}_{I^{(n)}}$. Note (thanks to the solidity of X) that $x^{(n)} \in X \cap \ell_0(I) \subset Y$ and that

$$|x_i| = \liminf_{n \rightarrow \infty} |x_i^{(n)}| \quad \forall i \in I.$$

Furthermore, boundedness of ι_0 and solidity of X imply $\|x^{(n)}\|_Y \leq \|\iota_0\| \cdot \|x^{(n)}\|_X \leq \|\iota_0\| \cdot \|x\|_X$ for all $n \in \mathbb{N}$. In particular, $(x^{(n)})_{n \in \mathbb{N}}$ is a bounded sequence in Y .

Using the Fatou property and the solidity of Y , we conclude $x \in Y$, with

$$\|x\|_Y = \|(x_i)_{i \in I}\|_Y \leq \liminf_{n \rightarrow \infty} \|x^{(n)}\|_Y \leq \|\iota_0\| \cdot \|x\|_X < \infty.$$

Thus, ι is well-defined and bounded with $\|\iota\| \leq \|\iota_0\|$. \square

Now, we analyze the existence of embeddings between a pair of nested sequence spaces. For this, we will *first* assume that the “inner index sets” $(I_k)_{k \in K}$ are disjoint. Furthermore, we assume that the *same* index sets are used on *both* sides of the embedding:

Lemma 4.7. *Let $X, Y \leq \mathbb{C}^K$ be two solid sequence spaces. Assume that for each $k \in K$, we are given a set $I^{(k)}$ and solid sequence spaces $X_k, Y_k \leq \mathbb{C}^{I^{(k)}}$. Finally, assume that the $(I^{(k)})_{k \in K}$ are pairwise disjoint and that the triangle constants for the spaces X_k and Y_k are uniformly bounded.*

Set $I := \biguplus_{k \in K} I^{(k)}$ and define the sequence $(\theta_k)_{k \in K} \in [0, \infty]^K$ by⁴

$$\theta_k := \begin{cases} \|\iota_k\| & \text{if } \delta_k \in X \text{ and } \iota_k : X_k \hookrightarrow Y_k \text{ is well-defined and bounded,} \\ 1, & \text{if } \delta_k \notin X, \\ \infty, & \text{otherwise.} \end{cases}$$

The following are equivalent:

- (1) *The embedding $\iota : X([X_k]_{k \in K}) \hookrightarrow Y([Y_k]_{k \in K})$ is well-defined and bounded.*
- (2) *We have $\theta_k < \infty$ for all $k \in K$ and the map⁵ $\gamma : X \rightarrow Y, (x_k)_{k \in K} \mapsto (\theta_k \cdot x_k)_{k \in K}$ is well-defined and bounded.*

In this case, we even have $\|\iota\| = \|\gamma\|$.

The implication (2) \Rightarrow (1), with $\|\iota\| \leq \|\gamma\|$ even holds if the $(I^{(k)})_{k \in K}$ are not pairwise disjoint. \blacktriangleleft

Proof. “(1) \Rightarrow (2)”: Let $k_0 \in K$. In case of $\delta_{k_0} \notin X$, $\theta_{k_0} = 1 < \infty$ is trivial. Thus, assume $\delta_{k_0} \in X$. Let $y = (y_i)_{i \in I^{(k_0)}} \in X_{k_0}$ be arbitrary and define

$$x_i := \begin{cases} 0, & \text{if } i \notin I^{(k_0)}, \\ y_i, & \text{if } i \in I^{(k_0)}. \end{cases}$$

Since the sets $(I^{(k)})_{k \in K}$ are pairwise disjoint, we get

$$\|(x_i)_{i \in I^{(k)}}\|_{X_k} = \begin{cases} 0 = \|y\|_{X_{k_0}} \cdot (\delta_{k_0})_k, & \text{if } k \neq k_0, \\ \|y\|_{X_{k_0}} = \|y\|_{X_{k_0}} \cdot (\delta_{k_0})_k, & \text{if } k = k_0. \end{cases}$$

Because of $\delta_{k_0} \in X$, this implies $x = (x_i)_{i \in I} \in X([X_k]_{k \in K})$ with

$$\|x\|_{X([X_k]_{k \in K})} = \left\| \left(\|y\|_{X_{k_0}} \cdot (\delta_{k_0})_k \right)_{k \in K} \right\|_X = \|y\|_{X_{k_0}} \cdot \|\delta_{k_0}\|_X < \infty.$$

Since ι is well-defined and bounded, we get $x \in Y([Y_k]_{k \in K})$. Furthermore, Lemma 3.8 shows that each of the coordinate maps $Y \rightarrow \mathbb{C}, (z_k)_{k \in K} \mapsto z_{k_0}$ is bounded. Hence,

$$\begin{aligned} \|y\|_{Y_{k_0}} &= \|(x_i)_{i \in I^{(k_0)}}\|_{Y_{k_0}} \lesssim_{k_0} \left\| \left(\|(x_i)_{i \in I^{(k)}}\|_{Y_k} \right)_{k \in K} \right\|_Y \\ &= \|x\|_{Y([Y_k]_{k \in K})} \\ &\leq \|\iota\| \cdot \|x\|_{X([X_k]_{k \in K})} \\ &\leq \|\iota\| \cdot \|\delta_{k_0}\|_X \cdot \|y\|_{X_{k_0}}. \end{aligned}$$

This shows that ι_{k_0} is well-defined and bounded with $\theta_{k_0} = \|\iota_{k_0}\| \lesssim_{k_0} \|\iota\| \cdot \|\delta_{k_0}\|_X < \infty$.

Thus, it remains to show that γ is well-defined and bounded, with $\|\gamma\| \leq \|\iota\|$. To this end, let $\varepsilon \in (0, 1)$ be arbitrary and let $x = (x_k)_{k \in K} \in X$. For each $k \in K$ with $x_k \neq 0$, we have $\delta_k \in X$ by solidity. By what we just showed, this implies that ι_k is well-defined and bounded. Thus (by definition of the operator norm), there is some sequence $y^{(k)} = (y_i^{(k)})_{i \in I^{(k)}}$ with $\|y^{(k)}\|_{X_k} = 1$ and such that

⁴In case of $\delta_k \notin X$, one can choose $\theta_k \in (0, \infty)$ arbitrarily; the statement of the theorem still holds. We only make the choice $\theta_k = 1$ for definiteness.

⁵Boundedness of γ is essentially the same as boundedness of the embedding $X \hookrightarrow Y_\theta$. The only drawback of this formulation is that Y_θ is in general not a well-defined sequence space, if $\theta_k = 0$ for some $k \in K$. This happens if and only if $\delta_k \in X$ and $I^{(k)} = \emptyset$. In particular, if $I^{(k)} \neq \emptyset$ for all $k \in K$, the two formulations are equivalent.

$\|y^{(k)}\|_{Y_k} \geq (1 - \varepsilon) \|\iota_k\| = (1 - \varepsilon) \theta_k$. For $k \in K$ with $x_k = 0$, we set $y^{(k)} := 0 \in X_k \cap Y_k \leq \mathbb{C}^{I^{(k)}}$. Note that we always have

$$|x_k| \cdot \|y^{(k)}\|_{Y_k} \geq \begin{cases} |x_k| \cdot (1 - \varepsilon) \theta_k, & \text{if } x_k \neq 0, \\ 0 = |x_k| \cdot (1 - \varepsilon) \theta_k, & \text{if } x_k = 0. \end{cases} \quad (4.1)$$

Now, define a sequence $z = (z_i)_{i \in I}$ by

$$z_i := x_k \cdot y_i^{(k)} \text{ for the unique } k = k_i \in K \text{ with } i \in I^{(k)}.$$

Note that k is uniquely determined since $I = \biguplus_{k \in K} I^{(k)}$, by assumption. Again by disjointness of the $(I^{(k)})_{k \in K}$, we have for $k \in K$ that

$$\|(z_i)_{i \in I^{(k)}}\|_{X_k} = \|(x_k \cdot y_i^{(k)})_{i \in I^{(k)}}\|_{X_k} = |x_k| \cdot \|y^{(k)}\|_{X_k} = \begin{cases} 0 = |x_k|, & \text{if } x_k = 0, \\ |x_k|, & \text{if } x_k \neq 0. \end{cases}$$

By solidity of X and since $x = (x_k)_{k \in K} \in X$, we conclude $z \in X([X_k]_{k \in K})$.

Since ι is well-defined and bounded, we get $z \in Y([Y_k]_{k \in K})$, with

$$\|z\|_{Y([Y_k]_{k \in K})} \leq \|\iota\| \cdot \|x\|_{X([X_k]_{k \in K})} = \|\iota\| \cdot \|(|x_k|)_{k \in K}\|_X = \|\iota\| \cdot \|x\|_X.$$

But disjointness of $(I^{(k)})_{k \in K}$ finally implies

$$\|(z_i)_{i \in I^{(k)}}\|_{Y_k} = \|(x_k \cdot y_i^{(k)})_{i \in I^{(k)}}\|_{Y_k} = |x_k| \cdot \|y^{(k)}\|_{Y_k} \stackrel{\text{Eq. (4.1)}}{\geq} (1 - \varepsilon) \cdot \theta_k |x_k|$$

and thus—by solidity of Y —that $(1 - \varepsilon) \cdot (\theta_k x_k)_{k \in K} \in Y$, with

$$\begin{aligned} (1 - \varepsilon) \|\gamma(x)\|_Y &= (1 - \varepsilon) \|(\theta_k |x_k|)_{k \in K}\|_Y \\ &\leq \left\| \left[\|(z_i)_{i \in I^{(k)}}\|_{Y_k} \right]_{k \in K} \right\|_Y \\ &= \|z\|_{Y([Y_k]_{k \in K})} \leq \|\iota\| \cdot \|x\|_X < \infty. \end{aligned}$$

Since $\varepsilon \in (0, 1)$ was arbitrary, we get $\gamma(x) \in Y$ and $\|\gamma(x)\|_Y \leq \|\iota\| \cdot \|x\|_X$. But $x \in X$ was arbitrary, so that we are done.

“(2) \Rightarrow (1)”: Here, we do *not* assume $(I^{(k)})_{k \in K}$ to be pairwise disjoint. Let $x = (x_i)_{i \in I} \in X([X_k]_{k \in K})$ be arbitrary. By definition, $x^{(k)} := (x_i)_{i \in I^{(k)}} \in X_k$ for each $k \in K$ and the sequence $y = (y_k)_{k \in K}$ defined by $y_k := \|x^{(k)}\|_{X_k}$ satisfies $y \in X$ and $\|y\|_X = \|x\|_{X([X_k]_{k \in K})}$. Hence—since γ is well-defined and bounded—the sequence $z := \gamma(y) = (\theta_k \cdot y_k)_{k \in K}$ satisfies $z \in Y$ and

$$\|z\|_Y \leq \|\gamma\| \cdot \|y\|_X = \|\gamma\| \cdot \|x\|_{X([X_k]_{k \in K})}.$$

Now, by assumption, we have $\theta_k < \infty$ for all $k \in K$. Thus, for $k \in K$, there are two cases:

Case 1. $\delta_k \in X$. In this case, $\theta_k < \infty$ implies that ι_k is well-defined and bounded, so that we get $x^{(k)} \in Y_k$ with

$$\|x^{(k)}\|_{Y_k} \leq \|\iota_k\| \cdot \|x^{(k)}\|_{X_k} = \theta_k \cdot y_k = z_k.$$

Case 2. $\delta_k \notin X$. In this case, solidity of X —together with $y \in X$ —yields $\|x^{(k)}\|_{X_k} = y_k = 0$ and hence $x^{(k)} = 0$, so that $x^{(k)} \in Y_k$ and

$$\|x^{(k)}\|_{Y_k} = 0 \leq z_k$$

hold trivially.

Using $z \in Y$, the estimate from the case distinction and solidity of Y , we finally get $(\|x^{(k)}\|_{Y_k})_{k \in K} \in Y$ and

$$\|x\|_{Y([Y_k]_{k \in K})} = \left\| (\|x^{(k)}\|_{Y_k})_{k \in K} \right\|_Y \leq \|z\|_Y \leq \|\gamma\| \cdot \|x\|_{X([X_k]_{k \in K})}.$$

Hence, ι is well-defined and bounded with $\|\iota\| \leq \|\gamma\|$. \square

The most common type of solid sequence spaces that we will consider are the weighted Lebesgue spaces ℓ_w^q . Thus, the next result is a valuable addition to the preceding lemma.

Lemma 4.8. (cf. [24, Lemma 5.1]) Let I be a set, let $p, q \in (0, \infty]$ and let $w = (w_i)_{i \in I}$ and $v = (v_i)_{i \in I}$ be two weights on I . Then, $\iota : \ell_w^q(I) \hookrightarrow \ell_v^p(I)$ is well-defined and bounded if and only if

$$C := \|(v_i/w_i)_{i \in I}\|_{\ell^{p \cdot (q/p)'}} is finite.$$

In this case, $\|\iota\| = C$. Here, we use the convention

$$p \cdot (q/p)' = \infty \text{ if } p = \infty \text{ or } q < p < \infty.$$

In the remaining case, where simultaneously $p \leq q$ and $p < \infty$, the expression $p \cdot (q/p)'$ is to be evaluated using the usual rules for computing the conjugate exponent and for calculation in $(0, \infty]$. \blacktriangleleft

Remark. We have

$$\frac{1}{p \cdot (q/p)'} = \left(\frac{1}{p} - \frac{1}{q} \right)_+. \quad (4.2)$$

In case of $p = \infty$, this is clear since the left-hand side is 0, while for the right-hand side, we have $\frac{1}{p} - \frac{1}{q} = -\frac{1}{q} \leq 0$ and hence $(p^{-1} - q^{-1})_+ = 0$. Likewise, for $q < p < \infty$, we have $p^{-1} < q^{-1}$ and thus $(p^{-1} - q^{-1})_+ = 0$ and the left-hand side of equation (4.2) vanishes as well.

Finally, if $p \leq q$ and $p < \infty$, then

$$\frac{1}{p \cdot (q/p)'} = \frac{1}{p} \cdot \left(1 - \frac{1}{q/p} \right) = \frac{1}{p} \cdot \left(1 - \frac{p}{q} \right) = \frac{1}{p} - \frac{1}{q} = \left(\frac{1}{p} - \frac{1}{q} \right)_+.$$

In particular, equation (4.2) yields

$$p \cdot (q/p)' < \infty \iff \frac{1}{p \cdot (q/p)'} > 0 \iff \frac{1}{p} - \frac{1}{q} > 0 \iff q > p. \quad (4.3) \quad \blacklozenge$$

Proof of Lemma 4.8. “ \Leftarrow ”: Let $x = (x_i)_{i \in I} \in \ell_w^q$ be arbitrary. Let us first assume $p < \infty$ and $p \leq q$. In this case, we have $r := q/p \in [1, \infty]$, so that we can apply Hölder’s inequality in the calculation

$$\begin{aligned} \|x\|_{\ell_v^p}^p &= \sum_{i \in I} (v_i \cdot |x_i|)^p \\ &= \sum_{i \in I} \left(\frac{v_i}{w_i} \right)^p \cdot (w_i \cdot |x_i|)^p \\ &\leq \|((v_i/w_i)^p)_{i \in I}\|_{\ell^{r'}} \cdot \|((w_i \cdot |x_i|)^p)_{i \in I}\|_{\ell^r} \\ &\stackrel{(*)}{=} \|(v_i/w_i)_{i \in I}\|_{\ell^{p \cdot (q/p)'}}^p \cdot \|(w_i \cdot x_i)_{i \in I}\|_{\ell^q}^p \\ &= C^p \cdot \|x\|_{\ell_w^q}^p < \infty. \end{aligned}$$

Here, the step marked with $(*)$ is a direct consequence of the definition of the ℓ^s -norm. A moment’s thought shows that this step is also valid in case of $(q/p)' = \infty$ or $q/p = \infty$.

Finally, taking p th roots of the above estimate completes the proof for $p < \infty$ and $p \leq q$.

Next, assume $p = \infty$. In this case, we have $p \cdot (q/p)' = \infty$ and hence $v_i/w_i \leq C$ for all $i \in I$. But this implies

$$|v_i \cdot x_i| = \frac{v_i}{w_i} \cdot |w_i \cdot x_i| \leq \frac{v_i}{w_i} \cdot \|x\|_{\ell_w^q} \leq C \cdot \|x\|_{\ell_w^q} < \infty$$

for each $i \in I$, which easily yields the claim.

Finally, assume $q < p < \infty$. Again, $p \cdot (q/p)' = \infty$, so that $v_i \leq C \cdot w_i$ for all $i \in I$. Note that we have a (quasi)-norm decreasing embedding $\ell^q \hookrightarrow \ell^p$. All in all, we get

$$\|x\|_{\ell_v^p} \leq C \cdot \|x\|_{\ell_w^p} \leq C \cdot \|x\|_{\ell_w^q} < \infty,$$

as desired.

“ \Rightarrow ”: First, let $i \in I$ be arbitrary. We have

$$v_i = \|\delta_i\|_{\ell_v^p} \leq \|\iota\| \cdot \|\delta_i\|_{\ell_w^q} = \|\iota\| \cdot w_i$$

and hence $v_i/w_i \leq \|\iota\|$. This proves the claim in all cases where $p \cdot (q/p)' = \infty$.

Thus, we can assume in the following that $p \leq q$ and $p < \infty$ hold. In this case, we have $p < \infty$ and $r := q/p \in [1, \infty]$. Let $\theta_i := (v_i/w_i)^p$ for $i \in I$ and note $p \cdot (q/p)' = p \cdot r'$, so that we get

$$C = \|(v_i/w_i)_{i \in I}\|_{\ell^{p \cdot (q/p)'}} = \|\theta\|_{\ell^{r'}}^{1/p}.$$

It is thus sufficient to show $\|\theta\|_{\ell^{r'}} \leq \|\iota\|^p$. But since $r, r' \in [1, \infty]$, it is well-known that

$$\|\theta\|_{\ell^{r'}} = \sup_{\substack{x=(x_i)_{i \in I} \in \ell_0(I) \\ \|x\|_{\ell^r} \leq 1}} \sum_{i \in I} |x_i \theta_i|,$$

where $\ell_0(I)$ denotes the space of finitely supported sequences over I .

Thus, let $x = (x_i)_{i \in I} \in \ell_0(I)$ with $\|x\|_{\ell^r} \leq 1$ be arbitrary. Define $y_i := w_i^{-1} \cdot |x_i|^{1/p}$ for $i \in I$ and note

$$\|y\|_{\ell_w^q} = \left\| \left(|x_i|^{1/p} \right)_{i \in I} \right\|_{\ell^q} = \|x\|_{\ell^{q/p}}^{1/p} = \|x\|_{\ell^r}^{1/p} \leq 1.$$

Since the embedding ι is well-defined and bounded and because of $p < \infty$, this implies

$$\left[\sum_{i \in I} |x_i \theta_i| \right]^{1/p} = \left[\sum_{i \in I} (v_i/w_i)^p \cdot |x_i| \right]^{1/p} = \left\| \frac{v_i}{w_i} \cdot |x_i|^{1/p} \right\|_{\ell^p} = \|y\|_{\ell_v^p} \leq \|\iota\|.$$

All in all, we have shown $\sum_{i \in I} |x_i \theta_i| \leq \|\iota\|^p$ for every finitely supported sequence $x = (x_i)_{i \in I} \in \ell_0(I)$ with $\|x\|_{\ell^r} \leq 1$. As seen above, this yields $\|\theta\|_{\ell^{r'}} \leq \|\iota\|^p$, which then implies the claim. \square

The preceding results yield a satisfactory characterization of existence of an embedding between two nested sequence spaces if the underlying sets $(I^{(k)})_{k \in K}$ are disjoint (and the same on both sides of the embedding). It is thus of interest to have a criterion which allows to reduce to this case. Such a criterion is given in the next lemma. Although the given criterion might appear rather technical, we will see later that it is useful and readily applicable in a number of situations, cf. Lemma 4.11 and Corollaries 5.8 and 5.12.

Lemma 4.9. *Let K be an index set and assume that for each $k \in K$, we are given a set $I^{(k)}$ and a weight $u_k = (u_{k,i})_{i \in I^{(k)}}$. Let $q \in (0, \infty]$.*

Let $I := \bigcup_{k \in K} I^{(k)}$ and define a relation \sim on K by

$$k \sim \ell \quad :\Longleftrightarrow \quad I^{(k)} \cap I^{(\ell)} \neq \emptyset.$$

*Let $X \leq \mathbb{C}^K$ be a solid sequence space for which the **generalized clustering map***

$$\Theta : X \rightarrow X, x = (x_k)_{k \in K} \mapsto \left[\sum_{\ell \in K \text{ with } \ell \sim k} x_\ell \right]_{k \in K} \quad (4.4)$$

is well-defined and bounded.

Furthermore, assume that we have a uniform bound

$$N := \sup_{k \in K} |[k]| < \infty \quad \text{where} \quad [k] := \{\ell \in K \mid \ell \sim k\} \quad (4.5)$$

and that there is some constant $C_u > 0$ with $u_{k,i} \leq C_u \cdot u_{\ell,i}$ for all $k, \ell \in K$ and $i \in I^{(k)} \cap I^{(\ell)}$.

Finally, assume that for each $k \in K$, we are given a subset $I^{(k,)} \subset I^{(k)}$ with $I = \bigcup_{k \in K} I^{(k,*)}$. Then*

$$\|x\|_{X([\ell_u^q(I^{(k)})]_{k \in K})} \geq \|x\|_{X([\ell_u^q(I^{(k,*)})]_{k \in K})} \geq C^{-1} \cdot \|x\|_{X([\ell_u^q(I^{(k)})]_{k \in K})}$$

for all sequences $x \in \mathbb{C}^I$ and some (fixed) constant $C = C(\|\Theta\|, C_u, N, q)$.

In particular, for every $x \in \mathbb{C}^I$, we have $x \in X([\ell_u^q(I^{(k)})]_{k \in K}) \iff x \in X([\ell_u^q(I^{(k,)})]_{k \in K})$. \blacktriangleleft*

Proof. Since X is solid, the estimate $\|x\|_{X([\ell_u^q(I^{(k)})]_{k \in K})} \geq \|x\|_{X([\ell_u^q(I^{(k,*)})]_{k \in K})}$ is a direct consequence of the estimate $\|y\|_{\ell_u^q(I^{(k,*)})} \leq \|y\|_{\ell_u^q(I^{(k)})}$ for all $y \in \mathbb{C}^{I^{(k,*)}}$, which in turn follows from $I^{(k,*)} \subset I^{(k)}$.

It is thus sufficient to show

$$\|x\|_{X([\ell_u^q(I^{(k)})]_{k \in K})} \leq C \cdot \|x\|_{X([\ell_u^q(I^{(k,*)})]_{k \in K})} \quad \text{assuming} \quad x \in X([\ell_u^q(I^{(k,*)})]_{k \in K}).$$

To this end, let $k \in K$ be arbitrary. For each $i \in I^{(k)} \subset I$, there is—because of $I = \bigcup_{\ell \in K} I^{(\ell, \star)}$ —some $\ell_i \in K$ with $i \in I^{(\ell_i, \star)}$. Note that $i \in I^{(k)} \cap I^{(\ell_i, \star)} \subset I^{(k)} \cap I^{(\ell_i)}$, so that $\ell_i \in [k]$. Furthermore, $u_{k,i} \leq C_u \cdot u_{\ell_i,i}$ as long as $i \in I^{(k)} \cap I^{(\ell_i, \star)} \subset I^{(k)} \cap I^{(\ell_i)}$. All in all, we get—in case of $q < \infty$ —that

$$\begin{aligned} \|x\|_{\ell_u^q(I^{(k)})}^q &= \sum_{i \in I^{(k)}} (u_{k,i} |x_i|)^q \leq \sum_{i \in I^{(k)}} \left(u_{k,i} \sum_{\ell \in [k]} [\mathbb{1}_{I^{(\ell, \star)}}(i) \cdot |x_i|] \right)^q \\ &\leq C_u^q \cdot \sum_{i \in I^{(k)}} \left(\sum_{\ell \in [k]} [\mathbb{1}_{I^{(\ell, \star)}}(i) \cdot u_{\ell,i} \cdot |x_i|] \right)^q. \end{aligned}$$

Now, note the general estimate

$$\left(\sum_{j \in J} \alpha_j \right)^q \leq (|J| \cdot \max \{ \alpha_j \mid j \in J \})^q \leq |J|^q \cdot \sum_{j \in J} \alpha_j^q \quad (4.6)$$

for finite sets J and arbitrary sequences $(\alpha_j)_{j \in J}$ of nonnegative numbers. Applied to the above estimate, we conclude

$$\begin{aligned} \|x\|_{\ell_u^q(I^{(k)})}^q &\leq C_u^q [k]^q \cdot \sum_{i \in I^{(k)}} \sum_{\ell \in [k]} (\mathbb{1}_{I^{(\ell, \star)}}(i) \cdot u_{\ell,i} \cdot |x_i|)^q \\ (\text{since } [k] \leq N) &\leq (C_u N)^q \cdot \sum_{\ell \in [k]} \sum_{i \in I^{(k)}} (\mathbb{1}_{I^{(\ell, \star)}}(i) \cdot u_{\ell,i} \cdot |x_i|)^q \\ &\leq (C_u N)^q \cdot \sum_{\ell \in [k]} \sum_{i \in I^{(\ell, \star)}} (u_{\ell,i} \cdot |x_i|)^q \\ &= (C_u N)^q \cdot \sum_{\ell \in [k]} \|x\|_{\ell_u^q(I^{(\ell, \star)})}^q \\ &\leq (C_u N)^q \cdot [k] \cdot \max \left\{ \|x\|_{\ell_u^q(I^{(\ell, \star)})}^q \mid \ell \in [k] \right\} \\ (\text{since } [k] \leq N) &\leq C_u^q N^{1+q} \cdot \left(\sum_{\ell \in [k]} \|x\|_{\ell_u^q(I^{(\ell, \star)})}^q \right)^q \\ &= C_u^q N^{1+q} \cdot [(\Theta y)_k]^q, \end{aligned}$$

where we defined

$$y_\ell := \|x\|_{\ell_u^q(I^{(\ell, \star)})} \quad \text{for } \ell \in K.$$

In case of $q = \infty$, we argue similarly: For arbitrary $i \in I^{(k)}$, we saw above that $i \in I^{(\ell_i, \star)}$ for some $\ell_i \in [k]$. Furthermore, $u_{k,i} \leq C_u \cdot u_{\ell_i,i}$ by our assumptions. Hence,

$$\begin{aligned} u_{k,i} \cdot |x_i| &\leq C_u \cdot \mathbb{1}_{I^{(\ell_i, \star)}}(i) \cdot u_{\ell_i,i} |x_i| \\ &\leq C_u \cdot \sum_{\ell \in [k]} \mathbb{1}_{I^{(\ell, \star)}}(i) \cdot u_{\ell,i} |x_i| \\ (\text{since } q = \infty) &\leq C_u \cdot \sum_{\ell \in [k]} \|x\|_{\ell_u^q(I^{(\ell, \star)})} \\ &= C_u \cdot (\Theta y)_k \end{aligned}$$

with $y = (y_\ell)_{\ell \in K}$ as above. Since $i \in I^{(k)}$ was arbitrary, we conclude

$$\|x\|_{\ell_u^q(I^{(k)})} \leq C_u \cdot (\Theta y)_k \leq C_u N^{1+\frac{1}{q}} \cdot (\Theta y)_k.$$

In summary, we have shown $0 \leq \|x\|_{\ell_u^q(I^{(k)})} \leq C_u N^{1+\frac{1}{q}} \cdot (\Theta y)_k$ for all $k \in K$ and arbitrary $q \in (0, \infty]$. But because of $x \in X \left([\ell_u^q(I^{(k, \star)})]_{k \in K} \right)$, we have $y \in X$ with $\|y\|_X = \|x\|_X \left([\ell_u^q(I^{(k, \star)})]_{k \in K} \right)$. By solidity of X and since we assume Θ to be well-defined and bounded, we conclude $x \in X \left([\ell_u^q(I^{(k)})]_{k \in K} \right)$

with

$$\begin{aligned}
 \|x\|_{X([\ell_u^q(I^{(k)})]_{k \in K})} &= \left\| \left(\|x\|_{\ell_u^q(I^{(k)})} \right)_{k \in K} \right\|_X \\
 &\leq C_u N^{1+\frac{1}{q}} \cdot \|\Theta y\|_X \\
 &\leq C_u N^{1+\frac{1}{q}} \|\Theta\| \cdot \|y\|_X \\
 &= C_u N^{1+\frac{1}{q}} \|\Theta\| \cdot \|x\|_{X([\ell_u^q(I^{(k,*)})]_{k \in K})} < \infty.
 \end{aligned}$$

This completes the proof. \square

The main point of the preceding lemma is that it allows to switch from the space $X([\ell_u^q(I^{(k)})]_{k \in K})$ to the slightly modified space $X([\ell_u^q(I^{(k,*)})]_{k \in K})$. In view of Lemma 4.7, it is preferable if the family $(I^{(k,*)})_{k \in K}$ can be chosen to be disjoint. Thus, the following lemma is helpful, since it shows—among other things—that one can always choose such a disjoint family $(I^{(k,*)})_{k \in K}$ which still satisfies $I = \bigcup_{k \in K} I^{(k,*)}$.

Lemma 4.10. *Under the assumptions of Lemma 4.9, the following are true:*

- (1) *For each pairwise disjoint family of sets $(I^{(k,0)})_{k \in K}$ with $I^{(k,0)} \subset I^{(k)}$, there is a pairwise disjoint family $(I^{(k,*)})_{k \in K}$ satisfying $I^{(k,0)} \subset I^{(k,*)} \subset I^{(k)}$ for all $k \in K$ and $I = \biguplus_{k \in K} I^{(k,*)}$.*
- (2) *There is a family $(I^{(k,n)})_{k \in K, n \in \underline{N}}$ with the following properties:*
 - (a) $I^{(k)} = \bigcup_{n=1}^N I^{(k,n)}$ for all $k \in K$,
 - (b) *For each $n \in \underline{N}$, we have $I = \biguplus_{k \in K} I^{(k,n)}$.*

Proof. 1. Set

$$\mathcal{A} := \left\{ (I_k)_{k \in K} \mid (I_k)_{k \in K} \text{ is pairwise disjoint and } I^{(k,0)} \subset I_k \subset I^{(k)} \text{ for all } k \in K \right\}.$$

Note that $(I^{(k,0)})_{k \in K} \in \mathcal{A}$, so that $\mathcal{A} \neq \emptyset$. Now, define a partial order on \mathcal{A} by setting

$$(I_k)_{k \in K} \leq (J_k)_{k \in K} \iff \forall k \in K : I_k \subset J_k.$$

It is not hard to see that \mathcal{A} is inductively ordered by “ \leq ”; indeed, if $((I_k^{(\alpha)})_{k \in K})_{\alpha \in A}$ is a (nonempty) chain in \mathcal{A} , then $I_k := \bigcup_{\alpha \in A} I_k^{(\alpha)}$ satisfies $I^{(k,0)} \subset I_k^{(\alpha)} \subset I_k \subset I^{(k)}$ for all $k \in K$ and arbitrary $\alpha \in A$. Furthermore, $(I_k)_{k \in K}$ is pairwise disjoint (and hence $(I_k)_{k \in K} \in \mathcal{A}$), since otherwise there are $k, \ell \in K$ with $k \neq \ell$ and some $x \in I_k \cap I_\ell$. By definition, there are thus $\alpha_1, \alpha_2 \in A$ with $x \in I_k^{(\alpha_1)} \cap I_\ell^{(\alpha_2)}$. But since we are considering a chain, we have $I_k^{(\alpha_1)} \subset I_k^{(\alpha_2)}$ and hence $x \in I_k^{(\alpha_2)} \cap I_\ell^{(\alpha_2)} = \emptyset$ or $I_\ell^{(\alpha_2)} \subset I_\ell^{(\alpha_1)}$ and hence $x \in I_k^{(\alpha_1)} \cap I_\ell^{(\alpha_1)} = \emptyset$. Both of these cases are absurd. Hence, $(I_k)_{k \in K} \in \mathcal{A}$ is an upper bound for the given chain.

By Zorn’s lemma, \mathcal{A} has a maximal element $(I^{(k,*)})_{k \in K}$. It remains to show $I = \bigcup_{k \in K} I^{(k,*)}$. If this is false, there is some $i \in I = \bigcup_{k \in K} I^{(k)}$ satisfying $i \notin \bigcup_{k \in K} I^{(k,*)}$. Choose $k_0 \in K$ with $i \in I^{(k_0)}$ and define

$$J^{(k,*)} := \begin{cases} I^{(k,*)}, & \text{if } k \neq k_0, \\ I^{(k,*)} \cup \{i\}, & \text{if } k = k_0. \end{cases}$$

It is then not hard to see $(J^{(k,*)})_{k \in K} \in \mathcal{A}$ and $(I^{(k,*)})_{k \in K} \leq (J^{(k,*)})_{k \in K} \not\leq (I^{(k,*)})_{k \in K}$, contradicting maximality of $(I^{(k,*)})_{k \in K}$.

2. Let $K_0 := \{k \in K \mid I^{(k)} \neq \emptyset\}$. The assumption of Lemma 4.9 implies that we have $|[k]| \leq N$ for all $k \in K_0$ and

$$[k] = \left\{ \ell \in K_0 \mid I^{(k)} \cap I^{(\ell)} \neq \emptyset \right\}.$$

Furthermore, the relation $k \sim \ell \iff I^{(k)} \cap I^{(\ell)} \neq \emptyset$ is easily seen to be reflexive and symmetric on K_0 . Thus, by Lemma 2.15, there is a finite partition $K_0 = \biguplus_{n=1}^N K_0^{(n)}$ such that $k \not\sim \ell$ for arbitrary $n \in \underline{N}$ and $k, \ell \in K_0^{(n)}$ with $k \neq \ell$.

Now, define

$$I_0^{(k,n)} := \begin{cases} I^{(k)}, & \text{if } k \in K_0^{(n)}, \\ \emptyset, & \text{if } k \notin K_0^{(n)} \end{cases}$$

for arbitrary $k \in K$ and $n \in \underline{N}$. We have $I_0^{(k,n)} \subset I^{(k)}$ for all $k \in K$. Furthermore, for $k, \ell \in K$ with $k \neq \ell$, we have $I_0^{(k,n)} \cap I_0^{(\ell,n)} = \emptyset$, unless possibly if $k, \ell \in K_0^{(n)}$. But in this case, we have $k \not\sim \ell$ by choice of $K_0^{(n)}$ and thus $I_0^{(k,n)} \cap I_0^{(\ell,n)} = I^{(k)} \cap I^{(\ell)} = \emptyset$.

By the first part, there is thus for each $n \in \underline{N}$ a pairwise disjoint family $(I^{(k,n)})_{k \in K}$ satisfying $I_0^{(k,n)} \subset I^{(k,n)} \subset I^{(k)}$ and $I = \biguplus_{k \in K} I^{(k,n)}$. It remains to check $I^{(k)} \subset \bigcup_{n=1}^N I^{(k,n)}$ for each $k \in K$. For $k \in K \setminus K_0$, this is clear since $I^{(k)} = \emptyset$. But for $k \in K_0$, we have $k \in K_0^{(n_k)}$ for some $n_k \in \underline{N}$ and hence $\bigcup_{n=1}^N I^{(k,n)} \supset I^{(k,n_k)} \supset I_0^{(k,n_k)} = I^{(k)}$, as desired. \square

Our next result indicates an important case in which the assumptions of Lemma 4.9 are satisfied:

Lemma 4.11. *Let $\mathcal{Q} = (Q_i)_{i \in I}$ be a family of subsets of \mathbb{R}^d and assume that $\mathcal{R} = (R_k)_{k \in K}$ is an admissible covering of a set $X \subset \mathbb{R}^d$.*

Furthermore, assume that \mathcal{Q} is almost subordinate to \mathcal{R} and pick some $n \in \mathbb{N}_0$. For each $k \in K$, choose a set

$$I^{(k)} \subset \{i \in I \mid Q_i \cap R_k^{n*} \neq \emptyset\}.$$

If we define a relation \sim on K by setting $k \sim \ell \iff I^{(k)} \cap I^{(\ell)} \neq \emptyset$, then $[k] := \{\ell \in K \mid \ell \sim k\}$ satisfies

$$[k] \subset k^{(2n+4k(\mathcal{Q}, \mathcal{R})+5)*} \quad \forall k \in K$$

and in particular

$$|[k]| \leq N_{\mathcal{R}}^{2n+4k(\mathcal{Q}, \mathcal{R})+5} \quad \forall k \in K. \quad \blacktriangleleft$$

Proof. Fix $k \in K$ and let $\ell \in [k]$. Hence, there is some $i \in I^{(k)} \cap I^{(\ell)}$, i.e. with $Q_i \cap R_k^{n*} \neq \emptyset \neq Q_i \cap R_\ell^{n*}$. This yields $k_0 \in k^{n*}$ and $\ell_0 \in \ell^{n*}$ with $Q_i \cap R_{k_0} \neq \emptyset \neq Q_i \cap R_{\ell_0}$. But in view of Lemma 2.11, this yields

$$\emptyset \neq Q_i \subset R_{k_0}^{(2k(\mathcal{Q}, \mathcal{R})+2)*} \cap R_{\ell_0}^{(2k(\mathcal{Q}, \mathcal{R})+2)*}.$$

In particular, $\ell_0 \in k_0^{(4k(\mathcal{Q}, \mathcal{R})+5)*} \subset k^{(n+4k(\mathcal{Q}, \mathcal{R})+5)*}$ and thus finally $\ell \in \ell_0^{n*} \subset k^{(2n+4k(\mathcal{Q}, \mathcal{R})+5)*}$.

All in all, we have shown $[k] \subset k^{(2n+4k(\mathcal{Q}, \mathcal{R})+5)*}$. The last part of the lemma is now a direct consequence of Lemma 2.9. \square

The following is a typical application of Lemmas 4.9, 4.8 and 4.10. It crucially uses that we have $\ell_u^r(I) = \ell^r\left([\ell_u^r(I^{(k)})]_{k \in K}\right)$ for every partition $I = \biguplus_{k \in K} I^{(k)}$.

Corollary 4.12. *Under the assumptions of Lemma 4.9, choose a family $(I^{(k, \mathfrak{h})})_{k \in K}$ with $I^{(k, \mathfrak{h})} \subset I^{(k)}$ for all $k \in K$ and with $I = \bigcup_{k \in K} I^{(k, \mathfrak{h})}$. Then the following are true:*

(1) *For a given weight $v = (v_i)_{i \in I}$ and $r \in (0, \infty]$, the embedding*

$$\iota : X([\ell_u^q(I^{(k)})]_{k \in K}) \hookrightarrow \ell_v^r(I)$$

satisfies

$$C^{-1} \cdot \|\gamma\| \leq \|\iota\| \leq \|\gamma^{\mathfrak{h}}\| \leq \|\gamma\|,$$

for some constant $C = C(\|\Theta\|, N, C_u, q, r)$ and

$$\begin{aligned} \gamma : X \rightarrow \ell^r(K), (x_k)_{k \in K} &\mapsto (\theta_k \cdot x_k)_{k \in K} \text{ with } \theta_k := \begin{cases} \|(v_i/u_{k,i})_{i \in I^{(k)}}\|_{\ell^{r \cdot (q/r)'}}, & \text{if } \delta_k \in X, \\ 1, & \text{else.} \end{cases} \\ \gamma^{\mathfrak{h}} : X \rightarrow \ell^r(K), (x_k)_{k \in K} &\mapsto (\theta_k^{\mathfrak{h}} \cdot x_k)_{k \in K} \text{ with } \theta_k^{\mathfrak{h}} := \begin{cases} \|(v_i/u_{k,i})_{i \in I^{(k, \mathfrak{h})}}\|_{\ell^{r \cdot (q/r)'}}, & \text{if } \delta_k \in X, \\ 1, & \text{else.} \end{cases} \end{aligned}$$

In particular, ι is well-defined and bounded if and only if $\theta_k < \infty$ for all $k \in K$ and if γ is well-defined and bounded.

(2) Let $C_X \geq 1$ be a triangle constant for X . For a given weight $v = (v_i)_{i \in I}$ and $r \in (0, \infty]$, the embedding

$$\iota : \ell_v^r(I) \hookrightarrow X([\ell_u^q(I^{(k)})]_{k \in K})$$

satisfies

$$C_1^{-1} \cdot \|\eta\| \leq \|\iota\| \leq C_2 \cdot \|\eta^\natural\| \leq C_2 \cdot \|\eta\|$$

for certain constants $C_1 = C_1(N, C_X, q, r)$, $C_2 = C_2(\|\Theta\|, C_u, N, q)$ and

$$\begin{aligned} \eta : \ell^r(K) &\rightarrow X, (x_k)_{k \in K} \mapsto (\vartheta_k \cdot x_k)_{k \in K} \text{ with } \vartheta_k := \|(u_{k,i}/v_i)_{i \in I^{(k)}}\|_{\ell^{q \cdot (r/q)'}} \\ \eta^\natural : \ell^r(K) &\rightarrow X, (x_k)_{k \in K} \mapsto (\vartheta_k^\natural \cdot x_k)_{k \in K} \text{ with } \vartheta_k^\natural := \|(u_{k,i}/v_i)_{i \in I^{(k, \natural)}}\|_{\ell^{q \cdot (r/q)'}} \end{aligned}$$

In particular, ι is well-defined and bounded if and only if $\vartheta_k < \infty$ for all $k \in K$ and if η is well-defined and bounded.

(3) In case of $X = \ell_w^s(K)$ for some weight $w = (w_k)_{k \in K}$, we have

$$\|\gamma^\natural\| = \left\| \left(w_k^{-1} \cdot \|(v_i/u_{k,i})_{i \in I^{(k, \natural)}}\|_{\ell^{r \cdot (q/r)'}} \right)_{k \in K} \right\|_{\ell^{s \cdot (s/r)'}}$$

and

$$\|\eta^\natural\| = \left\| \left(w_k \cdot \|(u_{k,i}/v_i)_{i \in I^{(k, \natural)}}\|_{\ell^{q \cdot (r/q)'}} \right)_{k \in K} \right\|_{\ell^{s \cdot (r/s)'}}.$$

In particular, γ^\natural or η^\natural is well-defined and bounded if and only if the respective right-hand side is finite. \blacktriangleleft

Proof. 1. “ \Rightarrow ”: First assume that ι is well-defined and bounded. By Lemma 4.10, there is for each $n \in \underline{N}$ a partition $I = \bigsqcup_{k \in K} I^{(k,n)}$ such that we have $I^{(k)} = \bigcup_{n=1}^N I^{(k,n)}$ for all $k \in K$. Now, Lemma 4.9 yields boundedness of $\iota^{(n)} : X([\ell_u^q(I^{(k,n)})]_{k \in K}) \hookrightarrow X([\ell_u^q(I^{(k)})]_{k \in K})$ for each $n \in \underline{N}$ and furthermore $\|\iota^{(n)}\| \leq C = C(\|\Theta\|, C_u, N, q)$.

Next, note that $I = \bigsqcup_{k \in K} I^{(k,n)}$ implies $\ell_v^r(I) = \ell^r([\ell_v^r(I^{(k,n)})]_{k \in K})$ with equal (quasi)-norms and $v_{k,i} = v_i$ for $k \in K$ and $i \in I^{(k,n)}$. Thus, by composition, we get boundedness of

$$\iota \circ \iota^{(n)} : X([\ell_u^q(I^{(k,n)})]_{k \in K}) \hookrightarrow \ell_v^r(I) = \ell^r([\ell_v^r(I^{(k,n)})]_{k \in K}).$$

In view of Lemmas 4.7 and 4.8, this yields $\theta_k^{(n)} < \infty$ for all $k \in K$, where

$$\theta_k^{(n)} = \begin{cases} \|\ell_u^q(I^{(k,n)}) \hookrightarrow \ell_v^r(I^{(k,n)})\| = \|(v_i/u_{k,i})_{i \in I^{(k,n)}}\|_{\ell^{r \cdot (q/r)'}} & , \text{ if } \delta_k \in X, \\ 1, & \text{ if } \delta_k \notin X, \end{cases}$$

and also boundedness of

$$\gamma^{(n)} : X \rightarrow \ell^r(K), (x_k)_{k \in K} \mapsto (\theta_k^{(n)} \cdot x_k)_{k \in K}$$

with

$$\|\gamma^{(n)}\| = \|\iota \circ \iota^{(n)}\| \leq C \cdot \|\iota\|.$$

Finally, we recall $I^{(k)} = \bigcup_{n=1}^N I^{(k,n)}$. Using the quasi-triangle inequality for $\ell^{r \cdot (q/r)'}$, this implies for $k \in K$ with $\delta_k \in X$ that

$$\theta_k \leq C_{r,q,N} \cdot \sum_{n=1}^N \theta_k^{(n)}.$$

If $\delta_k \notin X$, we trivially have $\theta_k = 1 \leq \sum_{n=1}^N \theta_k^{(n)}$. All in all, these considerations show (using the (quasi)-triangle inequality for ℓ^r) that

$$\begin{aligned} \|\gamma(x)\|_{\ell^r} &= \|(\theta_k \cdot x_k)_{k \in K}\|_{\ell^r} \\ &\leq C_{r,q,N} \cdot \left\| \left(\sum_{n=1}^N \theta_k^{(n)} \cdot |x_k| \right)_{k \in K} \right\|_{\ell^r} \\ &\leq C_{r,N} C_{r,q,N} \cdot \sum_{n=1}^N \left\| \left(\theta_k^{(n)} \cdot |x_k| \right)_{k \in K} \right\|_{\ell^r} \\ &\leq C_{r,N} C_{r,q,N} \cdot \left(\sum_{n=1}^N \|\gamma^{(n)}\| \right) \cdot \|x\|_X < \infty \end{aligned}$$

for all $x \in X$. Thus, γ is bounded with $\|\gamma\| \leq C \cdot N C_{r,N} C_{r,q,N} \cdot \|\iota\|$.

“ \Leftarrow ”: Because of $I^{(k,\natural)} \subset I^{(k)}$ for all $k \in K$, it is clear that if γ is bounded, then so is γ^\natural , with $\|\gamma^\natural\| \leq \|\gamma\|$. Thus, it suffices to assume that $\theta_k^\natural < \infty$ for all $k \in K$ and that γ^\natural is bounded. We need to show that ι is well-defined and bounded with $\|\iota\| \leq \|\gamma^\natural\|$.

The first part of Lemma 4.10 (applied to the family $(I^{(k,\natural)})_{k \in K}$ instead of $(I^{(k)})_{k \in K}$) yields a partition $I = \bigsqcup_{k \in K} I^{(k,\star)}$ with $I^{(k,\star)} \subset I^{(k,\natural)} \subset I^{(k)}$ for all $k \in K$. Since X is solid, we have a (quasi)-norm decreasing embedding

$$X([\ell_u^q(I^{(k)})]_{k \in K}) \hookrightarrow X([\ell_u^q(I^{(k,\star)})]_{k \in K}),$$

so that we have $\|\iota\| \leq \|\iota_1\|$ for

$$\iota_1 : X([\ell_u^q(I^{(k,\star)})]_{k \in K}) \xrightarrow{!} \ell_v^r(I) = \ell^r([\ell_v^r(I^{(k,\star)})]_{k \in K}).$$

Next, note that Lemma 4.8 yields

$$\tilde{\theta}_k := \|\ell_u^q(I^{(k,\star)}) \hookrightarrow \ell_v^r(I^{(k,\star)})\| = \|(v_i/u_{k,i})_{i \in I^{(k,\star)}}\|_{\ell^{r \cdot (q/r)'}} \leq \|(v_i/u_{k,i})_{i \in I^{(k,\natural)}}\|_{\ell^{r \cdot (q/r)'}} = \theta_k^\natural < \infty$$

for all $k \in K$ with $\delta_k \in X$. Thus, if we set $\tilde{\theta}_k := 1$ for those $k \in K$ with $\delta_k \notin X$, then Lemma 4.7 shows that ι_1 has the same norm as the map

$$\gamma_1 : X \rightarrow \ell^r(K), (x_k)_{k \in K} \mapsto (\tilde{\theta}_k \cdot x_k)_{k \in K}$$

But using $\tilde{\theta}_k \leq \theta_k^\natural$ for all $k \in K$, we finally get $\|\iota\| \leq \|\iota_1\| = \|\gamma_1\| \leq \|\gamma^\natural\| < \infty$, as desired.

2. “ \Rightarrow ”: Assume that ι is bounded. Lemma 4.10 yields for each $n \in \underline{N}$ a partition $I = \bigsqcup_{k \in K} I^{(k,n)}$ such that we have $I^{(k)} = \bigcup_{n=1}^N I^{(k,n)}$ for all $k \in K$. But because of $I^{(k,n)} \subset I^{(k)}$ and by solidity of X , it is not hard to see that we have a (quasi)-norm decreasing embedding

$$\varrho_n : X([\ell_u^q(I^{(k)})]_{k \in K}) \hookrightarrow X([\ell_u^q(I^{(k,n)})]_{k \in K})$$

for each $n \in \underline{N}$. By composition, we get boundedness of

$$\iota^{(n)} := \varrho_n \circ \iota : \ell_v^r(I) = \ell^r([\ell_v^r(I^{(k,n)})]_{k \in K}) \hookrightarrow X([\ell_u^q(I^{(k,n)})]_{k \in K}).$$

In view of Lemmas 4.7 and 4.8 (and since $\delta_k \in \ell^r$ for all $k \in K$ and since $(I^{(k,n)})_{k \in K}$ is pairwise disjoint), this yields boundedness of

$$\eta^{(n)} : \ell^r(K) \rightarrow X, (x_k)_{k \in K} \mapsto \left(\vartheta_k^{(n)} \cdot x_k \right)_{k \in K}$$

with $\|\eta^{(n)}\| = \|\iota^{(n)}\| \leq \|\iota\|$ and

$$\vartheta_k^{(n)} := \|\ell_v^r(I^{(k,n)}) \hookrightarrow \ell_u^q(I^{(k,n)})\| = \|(u_{k,i}/v_i)_{i \in I^{(k,n)}}\|_{\ell^{q \cdot (r/q)'}} < \infty \text{ for all } k \in K.$$

Now, using $I^{(k)} = \bigcup_{n=1}^N I^{(k,n)}$ for each $k \in K$, the (quasi)-triangle inequality for $\ell^{q \cdot (r/q)'}$ and Lemma 4.8 show

$$\vartheta_k = \|(u_{k,i}/v_i)_{i \in I^{(k)}}\|_{\ell^{q \cdot (r/q)'}} \leq C_{N,q,r} \cdot \sum_{n=1}^N \|(u_{k,i}/v_i)_{i \in I^{(k,n)}}\|_{\ell^{q \cdot (r/q)'}} = C_{N,q,r} \cdot \sum_{n=1}^N \vartheta_k^{(n)} < \infty$$

for all $k \in K$. All in all, using solidity of X and the quasi-triangle inequality for X , we get

$$\begin{aligned} \|(\vartheta_k \cdot x_k)_{k \in K}\|_X &\leq C_{N,q,r} \cdot \left\| \left(\sum_{n=1}^N \vartheta_k^{(n)} \cdot |x_k| \right)_{k \in K} \right\|_X \\ &\leq C_{N,C_X} C_{N,q,r} \cdot \sum_{n=1}^N \|\eta^{(n)}\|_X \\ &\leq C_{N,C_X} C_{N,q,r} \cdot \left(\sum_{n=1}^N \|\eta^{(n)}\| \right) \cdot \|x\|_{\ell^r} \\ &\leq C_{N,C_X} C_{N,q,r} \cdot N \|\iota\| \cdot \|x\|_{\ell^r} < \infty \end{aligned}$$

for all $x = (x_k)_{k \in K} \in \ell^r(K)$, so that η is well-defined and bounded with $\|\eta\| \leq C \cdot \|\iota\|$ for some constant $C = C(N, C_X, q, r)$.

“ \Leftarrow ”: It is clear that if η is bounded (and $\vartheta_k < \infty$ for all $k \in K$), then so is η^\natural , with $\|\eta^\natural\| \leq \|\eta\|$ (and with $\vartheta_k^\natural \leq \vartheta_k < \infty$ for all $k \in K$). Thus, it suffices to show that ι is bounded with $\|\iota\| \leq C \cdot \|\eta^\natural\|$, assuming that η^\natural is well-defined and bounded.

The first part of Lemma 4.10 (applied to the family $(I^{(k,\natural)})_{k \in K}$ instead of $(I^{(k)})_{k \in K}$) yields a partition $I = \biguplus_{k \in K} I^{(k,\star)}$ with $I^{(k,\star)} \subset I^{(k,\natural)}$ for all $k \in K$. In particular, we have

$$\infty > \vartheta_k^\natural = \|(u_{k,i}/v_i)_{i \in I^{(k,\natural)}}\|_{\ell^{q \cdot (r/q)'}} \geq \|(u_{k,i}/v_i)_{i \in I^{(k,\star)}}\|_{\ell^{q \cdot (r/q)'}} = \|\ell_v^r(I^{(k,\star)}) \hookrightarrow \ell_u^q(I^{(k,\star)})\| =: \widetilde{\vartheta}_k$$

for all $k \in K$, where the last step is justified by Lemma 4.8. Thus, by solidity of X , we see that boundedness of η^\natural implies boundedness of

$$\widetilde{\eta} : \ell^r(K) \rightarrow X, (x_k)_{k \in K} \mapsto (\widetilde{\vartheta}_k \cdot x_k)_{k \in K}$$

with $\|\widetilde{\eta}\| \leq \|\eta^\natural\|$.

But in view of Lemma 4.7, this yields boundedness of

$$\widetilde{\iota} : \ell_v^r(I) \stackrel{(*)}{=} \ell^r([\ell_v^r(I^{(k,\star)})]_{k \in K}) \hookrightarrow X([\ell_u^q(I^{(k,\star)})]_{k \in K}),$$

with $\|\widetilde{\iota}\| \leq \|\widetilde{\eta}\| \leq \|\eta^\natural\|$, where the (isometric(!)) identity marked with $(*)$ is a consequence of $I = \biguplus_{k \in K} I^{(k,\star)}$.

Finally, Lemma 4.9 shows that the identity map $\text{id} : X([\ell_u^q(I^{(k,\star)})]_{k \in K}) \hookrightarrow X([\ell_u^q(I^{(k)})]_{k \in K})$ is bounded with $\|\text{id}\| \leq C = C(\|\Theta\|, C_u, N, q)$. All in all, we see that $\iota = \text{id} \circ \widetilde{\iota}$ is bounded with $\|\iota\| \leq C \cdot \|\widetilde{\iota}\| \leq C \cdot \|\eta^\natural\|$, as claimed.

3. Note that we have $\delta_k \in X = \ell_w^s(K)$ for all $k \in K$, so that $\theta_k^\natural = \|(v_i/u_{k,i})_{i \in I^{(k,\natural)}}\|_{\ell^{r \cdot (q/r)'}}$ for all $k \in K$. In particular, $\theta_k^\natural = 0$ if and only if $I^{(k,\natural)} = \emptyset$ and likewise $\vartheta_k^\natural = 0$ if and only if $I^{(k,\natural)} = \emptyset$.

Thus, let $K_0 := \{k \in K \mid I^{(k,\natural)} \neq \emptyset\}$. It is then not hard to see that the embeddings

$$\gamma^{(0)} : \ell_w^s(K_0) \hookrightarrow \ell_{\theta^\natural}^r(K_0) \quad \text{and} \quad \eta^{(0)} : \ell^r(K_0) \hookrightarrow \ell_{w \cdot \vartheta^\natural}^s(K_0)$$

satisfy $\|\gamma^\natural\| = \|\gamma^{(0)}\|$ and $\|\eta^\natural\| = \|\eta^{(0)}\|$. But now, Lemma 4.8 shows

$$\|\gamma^{(0)}\| = \|(\theta_k^\natural/w_k)_{k \in K_0}\|_{\ell^{r \cdot (s/r)'}} = \|(\theta_k^\natural/w_k)_{k \in K}\|_{\ell^{r \cdot (s/r)'}}$$

and

$$\|\eta^{(0)}\| = \|(w_k \cdot \vartheta_k^\natural)_{k \in K_0}\|_{\ell^{s \cdot (r/s)'}} = \|(w_k \cdot \vartheta_k^\natural)_{k \in K}\|_{\ell^{s \cdot (r/s)'}}$$

as claimed. \square

We close this section by showing that \mathcal{Q} -regular sequence spaces again yield \mathcal{Q} -regular sequence spaces when weighted with \mathcal{Q} -moderate weights.

Lemma 4.13. *Let $\mathcal{Q} = (Q_i)_{i \in I}$ be a family of subsets of a set Y . If $X \leq \mathbb{C}^I$ is \mathcal{Q} -regular and if $u = (u_i)_{i \in I}$ is \mathcal{Q} -moderate, then X_u is \mathcal{Q} -regular, with*

$$\|\Gamma_{\mathcal{Q}}\|_{X_u \rightarrow X_u} \leq \|\Gamma_{\mathcal{Q}}\|_{X \rightarrow X} \cdot C_{u,\mathcal{Q}}.$$

In particular, $\ell_u^q(I)$ is \mathcal{Q} -regular with

$$\|\Gamma_{\mathcal{Q}}\|_{\ell_u^q \rightarrow \ell_u^q} \leq C_{u,\mathcal{Q}} \cdot N_{\mathcal{Q}}^{1+\frac{1}{q}},$$

for arbitrary $q \in (0, \infty]$. ◀

Proof. Let $x = (x_i)_{i \in I} \in X_u$ be arbitrary. We have

$$u_i \cdot |x_i^*| \leq \sum_{\ell \in i^*} u_i \cdot |x_\ell| \leq C_{u,\mathcal{Q}} \cdot \sum_{\ell \in i^*} u_\ell |x_\ell| = C_{u,\mathcal{Q}} \cdot (u \cdot |x|)_i^*.$$

By definition of X_u , we have $u \cdot x \in X$. By solidity of X , this implies $u \cdot |x| = |u \cdot x| \in X$ and thus also $(u \cdot |x|)^* \in X$, since X is \mathcal{Q} -regular. Again by solidity of X , we get $u \cdot |x^*| \in X$, with

$$\begin{aligned} \|x^*\|_{X_u} &= \|u \cdot x^*\|_X = \|u \cdot |x^*|\|_X \\ &\leq \|C_{u,\mathcal{Q}} \cdot (u \cdot |x|)^*\|_X \\ &\leq C_{u,\mathcal{Q}} \|\Gamma_{\mathcal{Q}}\|_{X \rightarrow X} \cdot \|u \cdot |x|\|_X \\ &= C_{u,\mathcal{Q}} \|\Gamma_{\mathcal{Q}}\|_{X \rightarrow X} \cdot \|u \cdot x\|_X \\ &= C_{u,\mathcal{Q}} \|\Gamma_{\mathcal{Q}}\|_{X \rightarrow X} \cdot \|x\|_{X_u} < \infty. \end{aligned}$$

This is the desired estimate.

For the second part, it suffices—in view of the first part—to show that $\ell^q(I)$ is \mathcal{Q} -regular with $\|\Gamma_{\mathcal{Q}}\|_{\ell^q \rightarrow \ell^q} \leq N_{\mathcal{Q}}^{1+\frac{1}{q}}$. Let us first assume $q < \infty$. In this case,

$$\left| \sum_{\ell \in i^*} x_\ell \right|^q \leq \left[\sum_{\ell \in i^*} |x_\ell| \right]^q \leq (|i^*| \cdot \max\{|x_\ell| : \ell \in i^*\})^q \leq N_{\mathcal{Q}}^q \cdot \sum_{\ell \in i^*} |x_\ell|^q.$$

All in all, we get

$$\|x^*\|_{\ell^q}^q \leq N_{\mathcal{Q}}^q \cdot \sum_{i \in I} \sum_{\ell \in i^*} |x_\ell|^q = N_{\mathcal{Q}}^q \cdot \sum_{\ell \in I} \left[|x_\ell|^q \cdot \sum_{i \in \ell^*} 1 \right] \leq N_{\mathcal{Q}}^{q+1} \cdot \|x\|_{\ell^q}^q < \infty,$$

which proves the desired estimate.

Finally, for $q = \infty$, simply note

$$|x_i^*| \leq \sum_{\ell \in i^*} |x_\ell| \leq |i^*| \cdot \|x\|_{\ell^\infty} \leq N_{\mathcal{Q}} \cdot \|x\|_{\ell^\infty}$$

and hence $\|x^*\|_{\ell^\infty} \leq N_{\mathcal{Q}} \cdot \|x\|_{\ell^\infty}$, as desired. ◻

5. SUFFICIENT CONDITIONS FOR EMBEDDINGS

In this section, we properly start to investigate the existence of embeddings between decomposition spaces. More precisely, we will derive *sufficient* conditions for the existence of such embeddings. As noted in the introduction, these conditions will take the form of embeddings between certain (nested) sequence spaces.

All results in this section will be based on the following two lemmata which investigate how the L^p -norm of a function f is related to the L^p -norm of its frequency localized parts $f_i = \mathcal{F}^{-1}(\varphi_i \cdot \widehat{f})$. Thanks to Plancherel's theorem and since the covering \mathcal{Q} is “almost disjoint”—because \mathcal{Q} is admissible—this is easy in case of $p = 2$. Our more general results will be derived from this basic case via interpolation.

Our first result shows how the L^p -norm of f can be estimated in terms of the L^p -norms of the f_i . This result is a simplified form of [24, Lemma 3.1] and similar to [23, Lemma 5.1.2]. Since it is crucial for the remainder of the paper, we nevertheless provide a proof.

Lemma 5.1. *Let $\mathcal{Q} = (Q_i)_{i \in I}$ be an L^1 -decomposition covering of the open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$. Furthermore, let $I_0 \subset I$ be finite, let $p \in (0, \infty]$ and $k \in \mathbb{N}_0$ and assume that for each $i \in I_0$, we are given some $f_i \in \mathcal{S}'(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with Fourier support $\text{supp } \widehat{f}_i \subset \overline{Q_i^{k*}}$.*

Then

$$\left\| \sum_{i \in I_0} f_i \right\|_{L^p} \leq C \cdot \left\| (\|f_i\|_{L^p})_{i \in I_0} \right\|_{\ell^{p \vee}}$$

for $C = C_{\mathcal{Q}, \Phi, 1} \cdot N_{\mathcal{Q}}^{2k+4}$, where $\Phi = (\varphi_i)_{i \in I}$ is some L^1 -BAPU for \mathcal{Q} . Here, $p^\nabla = \min\{p, p'\}$. \blacktriangleleft

Proof. We first handle the (easier) case $p \in (0, 1]$. In this case, we have $p' = \infty > p$ and hence $p^\nabla = p$. For $p \in (0, 1]$, it is well-known that $\|\cdot\|_{L^p}$ is a p -norm, i.e. we have $\|f + g\|_{L^p}^p \leq \|f\|_{L^p}^p + \|g\|_{L^p}^p$ for all measurable f, g . Indeed, this is an immediate consequence of the estimate

$$|a + b|^p \leq |a|^p + |b|^p$$

which holds for all $a, b \in \mathbb{C}$, since $p \in (0, 1]$. Using a straightforward induction, we thus get (since I_0 is finite) that

$$\left\| \sum_{i \in I_0} f_i \right\|_{L^p}^p \leq \sum_{i \in I_0} \|f_i\|_{L^p}^p = \left\| (\|f_i\|_{L^p})_{i \in I_0} \right\|_{\ell^{p^\nabla}}^p.$$

Taking p th roots completes the proof for $p \in (0, 1]$. In fact, we have shown that $C = 1$ is a possible choice in this case. Thus, we can assume $p \geq 1$ for the remainder of the proof.

Fix some L^1 -BAPU $\Phi = (\varphi_i)_{i \in I}$ for \mathcal{Q} . Our first aim is to show $\widehat{f}_i = \varphi_i^{(k+2)*} \widehat{f}_i$ for all $i \in I_0$. To this end, note that Lemma 2.4 implies $\varphi_i^{(k+1)*} \equiv 1$ on Q_i^{k*} and hence also $\varphi_i^{(k+1)*} \equiv 1$ on $\overline{Q_i^{k*}} \supset \text{supp } \widehat{f}_i$. Now, note that $\overline{Q_i^{k*}} \subset \left(Q_i^{(k+1)*}\right)^\circ =: U_i$, since we just showed $\overline{Q_i^{k*}} \subset \left(\varphi_i^{(k+1)*}\right)^{-1}(\mathbb{C}^*)$, which is an open(!) subset of $Q_i^{(k+1)*}$.

Finally, Lemma 2.4 shows $\varphi_i^{(k+2)*} \equiv 1$ on $Q_i^{(k+1)*} \supset U_i \supset \text{supp } \widehat{f}_i$. Hence, for $\psi \in C_c^\infty(\mathbb{R}^d)$, we have $\text{supp}(\psi - \varphi_i^{(k+2)*} \psi) \subset U_i^c \subset (\text{supp } \widehat{f}_i)^c$ and thus $\langle \widehat{f}_i, \psi \rangle = \langle \widehat{f}_i, \varphi_i^{(k+2)*} \psi \rangle = \langle \varphi_i^{(k+2)*} \widehat{f}_i, \psi \rangle$. Since $C_c^\infty(\mathbb{R}^d) \leq \mathcal{S}(\mathbb{R}^d)$ is dense, we have thus shown $\widehat{f}_i = \varphi_i^{(k+2)*} \widehat{f}_i$, as desired. Hence,

$$f_i = \mathcal{F}^{-1} \left(\varphi_i^{(k+2)*} \cdot \widehat{f}_i \right) \quad \text{for all } i \in I.$$

As a consequence, it suffices to show that the map

$$\Phi_p : \ell^{p^\nabla}(I_0; L^p(\mathbb{R}^d)) \rightarrow L^p(\mathbb{R}^d), (g_i)_{i \in I_0} \mapsto \sum_{i \in I_0} \mathcal{F}^{-1} \left(\varphi_i^{(k+2)*} \cdot \widehat{g}_i \right) = \sum_{i \in I_0} \left[\left(\mathcal{F}^{-1} \varphi_i^{(k+2)*} \right) * g_i \right]$$

is bounded for all $p \in [1, \infty]$, with $\|\Phi_p\| \leq C := C_{\mathcal{Q}, \Phi, 1} \cdot N_{\mathcal{Q}}^{2k+4}$, since this will imply

$$\begin{aligned} C \cdot \left\| (\|f_i\|_{L^p})_{i \in I_0} \right\|_{\ell^{p^\nabla}} &\geq \|\Phi_p\| \left\| (f_i)_{i \in I_0} \right\|_{\ell^{p^\nabla}(I_0; L^p(\mathbb{R}^d))} \\ &\geq \|\Phi_p((f_i)_{i \in I_0})\|_{L^p} \\ &= \left\| \sum_{i \in I_0} \mathcal{F}^{-1} \left(\varphi_i^{(k+2)*} \cdot \widehat{f}_i \right) \right\|_{L^p} = \left\| \sum_{i \in I_0} f_i \right\|_{L^p}, \end{aligned}$$

as desired.

Note that each Φ_p is well-defined, since I_0 is finite and since $L^p \rightarrow L^p, g \mapsto \left(\mathcal{F}^{-1} \varphi_i^{(k+2)*} \right) * g$ is well-defined and bounded as a consequence of Young's convolution inequality and since we have $\mathcal{F}^{-1} \varphi_i^{(k+2)*} \in L^1(\mathbb{R}^d)$ as a *finite* sum of L^1 -functions; in fact,

$$\left\| \mathcal{F}^{-1} \varphi_i^{(k+2)*} \right\|_{L^1} \leq \sum_{\ell \in i^{(k+2)*}} \left\| \mathcal{F}^{-1} \varphi_\ell \right\|_{L^1} \leq C_{\mathcal{Q}, \Phi, 1} \cdot |i^{(k+2)*}| \leq C_{\mathcal{Q}, \Phi, 1} \cdot N_{\mathcal{Q}}^{k+2}, \quad (5.1)$$

where the last step is justified by Lemma 2.9.

Let us first consider the case $p \in [1, 2]$. Here, $p^\nabla = p$ and hence $\Phi_p : \ell^p(I_0; L^p(\mathbb{R}^d)) \rightarrow L^p(\mathbb{R}^d)$. It is thus sufficient to show $\|\Phi_p\| \leq C$ for $p = 1$ and $p = 2$, since the general result for $p \in [1, 2]$ then follows by complex interpolation for vector-valued⁶ L^p -spaces (cf. [1, Theorems 5.1.1 and 5.1.2]).

⁶In this case, one can even use the usual classical Riesz-Thorin interpolation theorem (cf. [8, Theorem 6.27]) by identifying $\ell^p(I_0; L^p(\mathbb{R}^d))$ with $L^p(I_0 \times \mathbb{R}^d, \mu)$, where μ is the product measure of the counting measure on I_0 and Lebesgue measure on \mathbb{R}^d . This argument is not applicable, however, for $p \in [2, \infty]$, since the “outer” exponent p^∇ differs from the “inner” exponent p for the space $\ell^{p^\nabla}(I_0; L^p(\mathbb{R}^d))$.

The case $p = 1$ is easy: We simply use the triangle inequality for $L^1(\mathbb{R}^d)$ and Young's inequality for $L^1(\mathbb{R}^d)$ to get

$$\begin{aligned}
 \|\Phi_p((g_i)_{i \in I_0})\|_{L^1} &= \left\| \sum_{i \in I_0} \left(\mathcal{F}^{-1} \varphi_i^{(k+2)*} \right) * g_i \right\|_{L^1} \\
 &\leq \sum_{i \in I_0} \left\| \mathcal{F}^{-1} \varphi_i^{(k+2)*} \right\|_{L^1} \|g_i\|_{L^1} \\
 &\leq C_{\mathcal{Q}, \Phi, 1} N_{\mathcal{Q}}^{k+2} \cdot \sum_{i \in I_0} \|g_i\|_{L^1} \\
 &= C_{\mathcal{Q}, \Phi, 1} N_{\mathcal{Q}}^{k+2} \cdot \|(g_i)_{i \in I_0}\|_{\ell^1(I_0; L^1(\mathbb{R}^d))} \\
 &\leq C_{\mathcal{Q}, \Phi, 1} N_{\mathcal{Q}}^{2k+4} \cdot \|(g_i)_{i \in I_0}\|_{\ell^1(I_0; L^1(\mathbb{R}^d))}.
 \end{aligned}$$

Now, let us consider the case $p = 2$. Here, Plancherel's theorem implies

$$\begin{aligned}
 \|\Phi_p((g_i)_{i \in I_0})\|_{L^2}^2 &= \left\| \mathcal{F}^{-1} \left(\sum_{i \in I_0} \varphi_i^{(k+2)*} \widehat{g_i} \right) \right\|_{L^2}^2 \\
 &= \left\| \sum_{i \in I_0} \varphi_i^{(k+2)*} \cdot \widehat{g_i} \right\|_{L^2}^2 \\
 &= \int_{\mathbb{R}^d} \left| \sum_{i \in I_0} \varphi_i^{(k+2)*}(\xi) \cdot \widehat{g_i}(\xi) \right|^2 d\xi.
 \end{aligned}$$

Fix $\xi \in \mathbb{R}^d$ for the moment and employ the Cauchy-Schwarz inequality to derive

$$\begin{aligned}
 \left| \sum_{i \in I_0} \varphi_i^{(k+2)*}(\xi) \cdot \widehat{g_i}(\xi) \right|^2 &\leq \sum_{i \in I_0} \left| \varphi_i^{(k+2)*}(\xi) \right|^2 \cdot \sum_{i \in I_0} |\widehat{g_i}(\xi)|^2 \\
 &\leq C_{\mathcal{Q}, \Phi, 1}^2 \cdot N_{\mathcal{Q}}^{3k+7} \cdot \sum_{i \in I_0} |\widehat{g_i}(\xi)|^2.
 \end{aligned}$$

The last estimate is justified as follows: In case of $\xi \in \mathbb{R}^d \setminus \mathcal{O}$, it is trivially true, since this entails $\varphi_i^{(k+2)*}(\xi) = 0$ for all $i \in I \supset I_0$. In case of $\xi \in \mathcal{O}$, there is some $i_\xi \in I$ satisfying $\xi \in Q_{i_\xi}$. Hence, if $\varphi_i^{(k+2)*}(\xi) \neq 0$, we have $\xi \in Q_i^{(k+2)*} \cap Q_{i_\xi}$ and thus $i \in i_\xi^{(k+3)*}$. This implies

$$\begin{aligned}
 \sum_{i \in I_0} \left| \varphi_i^{(k+2)*}(\xi) \right|^2 &\leq \sum_{i \in i_\xi^{(k+3)*}} \left| \varphi_i^{(k+2)*}(\xi) \right|^2 \\
 &\stackrel{(*)}{\leq} \left| i_\xi^{(k+3)*} \right| \cdot (C_{\mathcal{Q}, \Phi, 1} \cdot N_{\mathcal{Q}}^{k+2})^2 \\
 &\leq C_{\mathcal{Q}, \Phi, 1}^2 \cdot N_{\mathcal{Q}}^{3k+7},
 \end{aligned}$$

where the last step used Lemma 2.9. Furthermore, the step marked with $(*)$ is justified by recalling equation (5.1) and noting $\left\| \varphi_i^{(k+2)*} \right\|_{L^\infty} \leq \left\| \mathcal{F}^{-1} \varphi_i^{(k+2)*} \right\|_{L^1}$, as a consequence of the Hausdorff-Young inequality.

All in all, we have shown

$$\begin{aligned}
 \|\Phi_p((g_i)_{i \in I_0})\|_{L^2}^2 &\leq C_{\mathcal{Q}, \Phi, 1}^2 \cdot N_{\mathcal{Q}}^{3k+7} \cdot \int_{\mathbb{R}^d} \sum_{i \in I_0} |\widehat{g}_i(\xi)|^2 \, d\xi \\
 &\leq (N_{\mathcal{Q}}^{2k+4} C_{\mathcal{Q}, \Phi, 1})^2 \cdot \sum_{i \in I_0} \|\widehat{g}_i\|_{L^2}^2 \\
 (\text{Plancherel}) &= (N_{\mathcal{Q}}^{2k+4} C_{\mathcal{Q}, \Phi, 1})^2 \cdot \sum_{i \in I_0} \|g_i\|_{L^2}^2 \\
 &= \left[N_{\mathcal{Q}}^{2k+4} C_{\mathcal{Q}, \Phi, 1} \cdot \|(g_i)_{i \in I_0}\|_{\ell^2(I_0; L^2(\mathbb{R}^d))} \right]^2,
 \end{aligned}$$

as desired. This completes the proof for $p \in [1, 2]$.

Finally, we consider the case $p \in [2, \infty]$. Here, $p^\nabla = p'$ and hence $\Phi_p : \ell^{p'}(I_0; L^p(\mathbb{R}^d)) \rightarrow L^p(\mathbb{R}^d)$. Note that every $p \in (2, \infty)$ satisfies $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{\infty} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}$ for some $\theta \in (0, 1)$ and $p_0 = \infty$, as well as $p_1 = 2$. Because of

$$\frac{1}{p'} = 1 - \frac{1}{p} = \theta \left(1 - \frac{1}{p_1}\right) + (1 - \theta) \left(1 - \frac{1}{p_0}\right) = \frac{\theta}{p_1'} + \frac{1-\theta}{p_0'} = \frac{\theta}{2} + \frac{1-\theta}{1},$$

we get

$$\ell^{p'}(I_0; L^p(\mathbb{R}^d)) = [\ell^2(I_0; L^2(\mathbb{R}^d)), \ell^1(I_0; L^\infty(\mathbb{R}^d))]_\theta,$$

where $[X, Y]_\theta$ denotes the space obtained by complex interpolation between X and Y with parameter $\theta \in [0, 1]$, cf. [1, Theorems 5.1.1 and 5.1.2]. Thus, it again suffices to prove $\|\Phi_p\| \leq C$ for $p = 2$ and $p = \infty$. But for $p = 2$, we already established it above, while proving the case $p \in [1, 2]$.

Finally, for $p = \infty$, we can argue just as for $p = 1$: The triangle inequality for L^∞ and Young's convolution inequality $L^1 * L^\infty \hookrightarrow L^\infty$ yield

$$\begin{aligned}
 \|\Phi_\infty((g_i)_{i \in I_0})\|_{L^\infty} &= \left\| \sum_{i \in I_0} (\mathcal{F}^{-1} \varphi_i^{(k+2)*}) * g_i \right\|_{L^\infty} \\
 &\leq \sum_{i \in I_0} \left\| \mathcal{F}^{-1} (\varphi_i^{(k+2)*}) \right\|_{L^1} \|g_i\|_{L^\infty} \\
 &\leq C_{\mathcal{Q}, \Phi, 1} \cdot N_{\mathcal{Q}}^{k+2} \cdot \sum_{i \in I_0} \|g_i\|_{L^\infty} \\
 &\leq C_{\mathcal{Q}, \Phi, 1} N_{\mathcal{Q}}^{2k+4} \cdot \|(g_i)_{i \in I_0}\|_{\ell^1(I_0; L^\infty(\mathbb{R}^d))}.
 \end{aligned}$$

Because of $\infty^\nabla = \infty' = 1$, this completes the proof. \square

The next lemma is concerned with a “converse” of Lemma 5.1, i.e. it shows how the L^p -norms of the individual “pieces” $f_i = \mathcal{F}^{-1}(\varphi_i \cdot \widehat{f})$ can be estimated in terms of the L^p -norm of f itself:

Lemma 5.2. *Let $\mathcal{Q} = (Q_i)_{i \in I}$ be an L^1 -decomposition covering of the open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ with L^1 -BAPU $\Phi = (\varphi_i)_{i \in I}$. Then, for each $p \in [1, \infty]$ and $k \in \mathbb{N}_0$, the map*

$$\Gamma_p^{(k)} : L^p(\mathbb{R}^d) \rightarrow \ell^{p^\Delta}(I; L^p(\mathbb{R}^d)), g \mapsto (\mathcal{F}^{-1}(\varphi_i^{k*} \cdot \widehat{g}))_{i \in I}$$

is well-defined and bounded with

$$\left\| \Gamma_p^{(k)} \right\| \leq C_{\mathcal{Q}, \Phi, 1} \cdot N_{\mathcal{Q}}^{2k+1}.$$

Here, $p^\Delta = \max\{p, p'\}$. \blacktriangleleft

Proof. As in the proof of Lemma 5.1, it suffices by complex interpolation to establish the claim for $p \in \{1, 2, \infty\}$. Furthermore, Lemma 2.9 shows

$$\left\| \mathcal{F}^{-1} \varphi_i^{k*} \right\|_{L^1} \leq \sum_{\ell \in i^{k*}} \left\| \mathcal{F}^{-1} \varphi_\ell \right\|_{L^1} \leq |i^{k*}| \cdot C_{\mathcal{Q}, \Phi, 1} \leq N_{\mathcal{Q}}^k C_{\mathcal{Q}, \Phi, 1},$$

so that Young's inequality implies for arbitrary $i \in I$, $p \in [1, \infty]$ and $g \in L^p(\mathbb{R}^d)$ that

$$\left\| \mathcal{F}^{-1}(\varphi_i^{k*} \cdot \widehat{g}) \right\|_{L^p} \leq \left\| \mathcal{F}^{-1} \varphi_i^{k*} \right\|_{L^1} \cdot \|g\|_{L^p} \leq N_{\mathcal{Q}}^k C_{\mathcal{Q}, \Phi, 1} \cdot \|g\|_{L^p} \leq N_{\mathcal{Q}}^{2k+1} C_{\mathcal{Q}, \Phi, 1} \cdot \|g\|_{L^p}.$$

Since we have $p^\Delta = \infty$ for $p \in \{1, \infty\}$, this yields the claim for $p = 1$ and $p = \infty$.

It remains to consider $p = 2$. Here, we have $p^\Delta = 2$. In view of Plancherel's theorem, this implies for arbitrary finite sets $I_0 \subset I$ that

$$\begin{aligned} \sum_{i \in I_0} \|\mathcal{F}^{-1}(\varphi_i^{k*} \cdot \widehat{g})\|_{L^2}^2 &= \sum_{i \in I_0} \int_{\mathbb{R}^d} |\varphi_i^{k*}(\xi) \cdot \widehat{g}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^d} |\widehat{g}(\xi)|^2 \cdot \sum_{i \in I_0} |\varphi_i^{k*}(\xi)|^2 d\xi. \end{aligned}$$

But for $\xi \in \mathbb{R}^d \setminus \mathcal{O}$, we have $\sum_{i \in I_0} |\varphi_i^{k*}(\xi)|^2 = 0$. If otherwise $\xi \in \mathcal{O}$, there is some $i_\xi \in I$ satisfying $\xi \in Q_{i_\xi}$. For each $i \in I_0$ with $\varphi_i^{k*}(\xi) \neq 0$, this implies $\emptyset \neq Q_{i_\xi} \cap Q_i^{k*}$ and hence $i \in i_\xi^{(k+1)*}$. But Lemma 2.9 and the Hausdorff-Young inequality imply

$$\|\varphi_i^{k*}\|_{L^\infty} \leq \sum_{\ell \in i^{k*}} \|\varphi_\ell\|_{L^\infty} \leq \sum_{\ell \in i^{k*}} \|\mathcal{F}^{-1}\varphi_\ell\|_{L^1} \leq |i^{k*}| C_{\mathcal{Q}, \Phi, 1} \leq C_{\mathcal{Q}, \Phi, 1} \cdot N_{\mathcal{Q}}^k,$$

so that we get

$$\sum_{i \in I_0} |\varphi_i^{k*}(\xi)|^2 \leq \sum_{i \in i_\xi^{(k+1)*}} \|\varphi_i^{k*}\|_{L^\infty}^2 \leq |i_\xi^{(k+1)*}| \cdot (C_{\mathcal{Q}, \Phi, 1} N_{\mathcal{Q}}^k)^2 \leq C_{\mathcal{Q}, \Phi, 1}^2 N_{\mathcal{Q}}^{3k+2}.$$

All in all, we have shown

$$\begin{aligned} \sum_{i \in I_0} \|\mathcal{F}^{-1}(\varphi_i^{k*} \cdot \widehat{g})\|_{L^2}^2 &\leq \int_{\mathbb{R}^d} |\widehat{g}(\xi)|^2 \cdot \sum_{i \in I_0} |\varphi_i^{k*}(\xi)|^2 d\xi \\ &\leq (C_{\mathcal{Q}, \Phi, 1} N_{\mathcal{Q}}^{2k+1} \cdot \|\widehat{g}\|_{L^2})^2 \\ (\text{Plancherel}) &= (C_{\mathcal{Q}, \Phi, 1} N_{\mathcal{Q}}^{2k+1} \cdot \|g\|_{L^2})^2. \end{aligned}$$

Since this holds for every finite subset $I_0 \subset I$, we finally arrive at

$$\left\| \Gamma_2^{(k)}(g) \right\|_{\ell^{2\Delta}(I; L^2(\mathbb{R}^d))} = \left(\sum_{i \in I} \|\mathcal{F}^{-1}(\varphi_i^{k*} \cdot \widehat{g})\|_{L^2}^2 \right)^{1/2} \leq C_{\mathcal{Q}, \Phi, 1} N_{\mathcal{Q}}^{2k+1} \cdot \|g\|_{L^2}$$

for all $g \in L^2(\mathbb{R}^d)$. As seen above, this completes the proof. \square

We want to allow embeddings of the form $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$, i.e. the possibility $p_1 \neq p_2$. To this end, the following lemma is crucial.

Lemma 5.3. *Let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ be a semi-structured admissible covering of the open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$.*

Let $k \in \mathbb{N}_0$ and $p_0, p_1, p_2 \in (0, \infty]$ with $p_0 \leq p_1 \leq p_2$. Then there is a constant $C = C(d, k, p_0, \mathcal{Q})$ (independent of p_1, p_2) such that

$$\|\mathcal{F}^{-1}(\gamma_i f)\|_{L^{p_2}} \leq C \cdot |\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot \|\mathcal{F}^{-1}(\gamma_i f)\|_{L^{p_1}}$$

holds for each distribution $f \in \mathcal{D}'(\mathcal{O})$, all $i \in I$ and each $\gamma_i \in C_c^\infty(\mathcal{O})$ with $\gamma_i \equiv 0$ on $\mathcal{O} \setminus Q_i^{k}$.* \blacktriangleleft

Proof. By Lemma 2.7, there is some $R = R(R_{\mathcal{Q}}, C_{\mathcal{Q}}, k) > 0$ such that $Q_i^{k*} \subset T_i(\overline{B_R}(0)) + b_i$ holds for all $i \in I$. By enlarging R , we can furthermore assume that $L := \lambda(B_1(0)) \cdot R^d \geq 1$. But note that this necessitates $R = R(R_{\mathcal{Q}}, C_{\mathcal{Q}}, k, d)$.

Fix a function $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\eta|_{\overline{B_R}(0)} \equiv 1$. Let $f \in \mathcal{D}'(\mathcal{O})$ and $i \in I$ with $\mathcal{F}^{-1}(\gamma_i f) \in L^{p_1}(\mathbb{R}^d)$ (otherwise, there is nothing to show).

Let us first consider the case $p_1 \in (0, 1)$. Here, Corollary 3.3 implies $\mathcal{F}^{-1}(\gamma_i f) \in L^{p_2}(\mathbb{R}^d)$ with

$$\begin{aligned} \|\mathcal{F}^{-1}(\gamma_i f)\|_{L^{p_2}} &\leq [\lambda(T_i(\overline{B_R}(0)) + b_i)]^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot \|\mathcal{F}^{-1}(\gamma_i f)\|_{L^{p_1}} \\ &= L^{p_1^{-1} - p_2^{-1}} \cdot |\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot \|\mathcal{F}^{-1}(\gamma_i f)\|_{L^{p_1}} \\ &\leq L^{p_0^{-1}} \cdot |\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot \|\mathcal{F}^{-1}(\gamma_i f)\|_{L^{p_1}}. \end{aligned}$$

Here, we used

$$\operatorname{supp} \left(\widehat{\mathcal{F}^{-1}(\gamma_i f)} \right) \subset \operatorname{supp} \gamma_i \subset \overline{Q_i^{k*}} \subset T_i \left(\overline{B_R}(0) \right) + b_i$$

in the first line. In the last line, we used $p_1^{-1} - p_2^{-1} \leq p_1^{-1} \leq p_0^{-1}$ and $L \geq 1$. This completes the proof in the case $p_1 \in (0, 1)$, since L only depends on d, k and R_Q, C_Q .

Now, let us assume $p_1 \in [1, \infty]$, which also entails $p_2 \in [1, \infty]$, because of $p_2 \geq p_1$. For $i \in I$, define

$$\eta_i : \mathbb{R}^d \rightarrow \mathbb{C}, \xi \mapsto \eta \left(T_i^{-1} (\xi - b_i) \right).$$

With this definition, we have $\eta_i \equiv 1$ on Q_i^{k*} ; indeed, note that $\xi \in Q_i^{k*} \subset T_i \left(\overline{B_R}(0) \right) + b_i$ entails $T_i^{-1} (\xi - b_i) \in \overline{B_R}(0)$ and thus $\eta_i (\xi) = 1$. Hence, $\eta_i \gamma_i = \gamma_i$ and furthermore

$$\mathcal{F}^{-1} (\gamma_i f) = \mathcal{F}^{-1} (\eta_i \cdot \gamma_i f) = (\mathcal{F}^{-1} \eta_i) * \mathcal{F}^{-1} (\gamma_i f).$$

Because of $1 \leq p_1 \leq p_2$, we see $-1 \leq -\frac{1}{p_1} \leq \frac{1}{p_2} - \frac{1}{p_1} \leq 0$ and hence $1 + \frac{1}{p_2} - \frac{1}{p_1} \in [0, 1]$. We can thus define $q \in [1, \infty]$ by $\frac{1}{q} = 1 + \frac{1}{p_2} - \frac{1}{p_1}$ with the understanding of $q = \infty$ in case of $1 + \frac{1}{p_2} - \frac{1}{p_1} = 0$. By the general form of Young's inequality (cf. [8, Proposition 8.9(a)]), we derive

$$\begin{aligned} \|\mathcal{F}^{-1} (\gamma_i f)\|_{L^{p_2}} &= \|(\mathcal{F}^{-1} \eta_i) * \mathcal{F}^{-1} (\gamma_i f)\|_{L^{p_2}} \\ &\leq \|\mathcal{F}^{-1} \eta_i\|_{L^q} \cdot \|\mathcal{F}^{-1} (\gamma_i f)\|_{L^{p_1}} < \infty, \end{aligned}$$

where we used $\eta_i \in C_c^\infty(\mathbb{R}^d)$ and thus $\mathcal{F}^{-1} \eta_i \in \mathcal{S}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$.

Since $\eta_i = L_{b_i}(\eta \circ T_i^{-1})$, a straightforward calculation using elementary properties of the Fourier transform (cf. [8, Theorem 8.22]) yields

$$\mathcal{F}^{-1} \eta_i = |\det T_i| \cdot M_{b_i}(\Delta_{T_i}[\mathcal{F}^{-1} \eta]),$$

where we recall the notation $\Delta_h f = f \circ h^T$. We conclude

$$\begin{aligned} \|\mathcal{F}^{-1} \eta_i\|_{L^q} &= |\det T_i| \cdot \|\Delta_{T_i}[\mathcal{F}^{-1} \eta]\|_{L^q} \\ &= |\det T_i|^{1-\frac{1}{q}} \cdot \|\mathcal{F}^{-1} \eta\|_{L^q} \\ &= |\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot \|\mathcal{F}^{-1} \eta\|_{L^q} \\ &\leq \max \{ \|\mathcal{F}^{-1} \eta\|_{L^1}, \|\mathcal{F}^{-1} \eta\|_{L^\infty} \} \cdot |\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}}. \end{aligned}$$

Here, the last step used the estimate $\|f\|_{L^q} \leq \max \{ \|f\|_{L^1}, \|f\|_{L^\infty} \}$ which is valid for all measurable $f : \mathbb{R}^d \rightarrow \mathbb{C}$ and $q \in [1, \infty]$, cf. [8, Proposition 6.10]. Further, recall that η only depends on d and on $R = R(d, k, R_Q, C_Q)$, so that $C = \max \{ \|\mathcal{F}^{-1} \eta\|_{L^1}, \|\mathcal{F}^{-1} \eta\|_{L^\infty} \}$ is of the form $C = C(d, k, p_0, Q)$, as desired. Actually, C does not depend on p_0 in this case. \square

As a corollary, we see that every L^p -BAPU is also an L^q -BAPU for all $q \geq p$.

Corollary 5.4. *Let $Q = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ be a semi-structured covering of $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ and let $\Phi = (\varphi_i)_{i \in I}$ be an L^p -BAPU for Q . Then, for every $q \in [p, \infty]$, Φ is also an L^q -BAPU for Q with*

$$C_{Q, \Phi, q} \leq C \cdot C_{Q, \Phi, p},$$

for some constant $C = C(d, Q, p)$. \blacktriangleleft

Proof. In case of $p \geq 1$ (and $q \geq p$), the definition for an L^q -BAPU is the same as that for an L^p -BAPU and $C_{Q, \Phi, q} = C_{Q, \Phi, p}$. Thus, we can assume $p \in (0, 1)$.

Lemma 5.3 (with $p_0 = p = p_1$, $p_2 = r$, $k = 0$, $f \equiv 1$ and $\gamma_i = \varphi_i$) yields a constant $C = C(d, Q, p)$ such that

$$\begin{aligned} \|\mathcal{F}^{-1} \varphi_i\|_{L^r} &= \|\mathcal{F}^{-1} (\varphi_i \cdot 1)\|_{L^r} \leq C \cdot |\det T_i|^{\frac{1}{p} - \frac{1}{r}} \cdot \|\mathcal{F}^{-1} (\varphi_i \cdot 1)\|_{L^p} \\ &\leq C \cdot |\det T_i|^{\frac{1}{p} - \frac{1}{r}} \cdot |\det T_i|^{1 - \frac{1}{p}} \cdot C_{Q, \Phi, p} \\ &= C C_{Q, \Phi, p} \cdot |\det T_i|^{1 - \frac{1}{r}} \end{aligned} \tag{5.2}$$

holds for all $i \in I$ and $r \geq p$. For $q \in [p, 1)$, this implies (for $r = q \geq p$) that

$$C_{Q, \Phi, q} = \sup_{i \in I} |\det T_i|^{\frac{1}{q} - 1} \|\mathcal{F}^{-1} \varphi_i\|_{L^q} \leq C \cdot C_{Q, \Phi, p} < \infty,$$

as desired. Finally, for $q \in [1, \infty]$, we apply equation (5.2) with $r = 1 \geq p$ to get

$$C_{\mathcal{Q}, \Phi, q} = \sup_{i \in I} \|\mathcal{F}^{-1} \varphi_i\|_{L^1} \leq C \cdot C_{\mathcal{Q}, \Phi, p} < \infty$$

as well. \square

Before we can state and prove our first sufficient criterion for embeddings between decomposition spaces, we need one final technical lemma.

Lemma 5.5. *Let $I, J \neq \emptyset$ be two index-sets and let $p, q \in (0, \infty]$ with $p \leq q$. Then, for an arbitrary sequence $(x_{i,j})_{(i,j) \in I \times J} \in \mathbb{C}^{I \times J}$, we have*

$$\left\| \left(\| (x_{i,j})_{i \in I} \|_{\ell^p} \right)_{j \in J} \right\|_{\ell^q} \leq \left\| \left(\| (x_{i,j})_{j \in J} \|_{\ell^q} \right)_{i \in I} \right\|_{\ell^p}. \quad \blacktriangleleft$$

Proof. Let us first assume $q < \infty$, which also implies $p < \infty$, since $p \leq q$. Then

$$\begin{aligned} \left\| \left(\| (x_{i,j})_{i \in I} \|_{\ell^p} \right)_{j \in J} \right\|_{\ell^q} &= \left(\sum_{j \in J} \left(\sum_{i \in I} |x_{i,j}|^p \right)^{q/p} \right)^{1/q} \\ &= \left\| \left(\sum_{i \in I} |x_{i,j}|^p \right)_{j \in J} \right\|_{\ell^{q/p}}^{1/p} \\ &\stackrel{(*)}{\leq} \left[\sum_{i \in I} \left\| (|x_{i,j}|^p)_{j \in J} \right\|_{\ell^{q/p}} \right]^{1/p} \\ &= \left(\sum_{i \in I} \left[\sum_{j \in J} (|x_{i,j}|^p)^{q/p} \right]^{p/q} \right)^{1/p} \\ &= \left(\sum_{i \in I} \left[\sum_{j \in J} |x_{i,j}|^q \right]^{p/q} \right)^{1/p} \\ &= \left\| \left(\| (x_{i,j})_{j \in J} \|_{\ell^q} \right)_{i \in I} \right\|_{\ell^p}, \end{aligned}$$

as desired. Here, the step marked with $(*)$ is justified by the triangle inequality for $\ell^{q/p}(J)$, since $q/p \geq 1$.

Now, we consider the case $q = \infty$. To this end, let $j \in J$ be arbitrary. Then, by solidity of ℓ^p ,

$$\| (x_{i,j})_{i \in I} \|_{\ell^p} = \| (|x_{i,j}|)_{i \in I} \|_{\ell^p} \leq \left\| \left(\sup_{j \in J} |x_{i,j}| \right)_{i \in I} \right\|_{\ell^p} \stackrel{q=\infty}{=} \left\| \left(\| (x_{i,j})_{j \in J} \|_{\ell^q} \right)_{i \in I} \right\|_{\ell^p}.$$

Since this holds for arbitrary $j \in J$, we get the desired inequality. \square

All of our sufficient conditions for the existence of embeddings between decomposition spaces will be based on the following theorem. Its assumptions are very general, but usually quite tedious to verify. Thus, in the remainder of this section, we will derive several more specialized consequences of this general theorem, whose assumptions can often be verified more easily.

Theorem 5.6. Let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J} = (S_j P'_j + c_j)_{j \in J}$ be two semi-structured admissible coverings of the open sets $\mathcal{O} \subset \mathbb{R}^d$ and $\mathcal{O}' \subset \mathbb{R}^d$, respectively. Let $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ be two solid sequence spaces which are \mathcal{Q} -regular and \mathcal{P} -regular, respectively.

Let $p_1, p_2 \in (0, \infty]$ with $p_1 \leq p_2$ and assume that \mathcal{Q} admits an L^{p_1} -BAPU $\Phi = (\varphi_i)_{i \in I}$.

Fix $I_0 \subset I$ and let $K \neq \emptyset$ be an index-set. For each $k \in K$, let $I^{(k)} \subset I_0$ and $J^{(k)} \subset J$ with

$$\left(\bigcup_{i \in I_0 \setminus I^{(k)}} Q_i \right) \cap \left(\bigcup_{j \in J^{(k)}} P_j \right) = \emptyset \quad (5.3)$$

for all $k \in K$. Let $J_0 \subset J_{00} := \bigcup_{k \in K} J^{(k)}$.

For each $k \in K$, choose some $q_k \in [p_1, p_2]$. Let $q^{(0)} := \inf_{k \in K} q_k$ and assume that \mathcal{P} admits an $L^{q^{(0)}}$ -BAPU $\Psi = (\psi_j)_{j \in J}$.

Let $X \leq \mathbb{C}^K$ be a solid sequence space on K and define two weights $v = (v_{k,i})_{k \in K, i \in I^{(k)}}$ and $w = (w_{k,j})_{k \in K, j \in J^{(k)}}$ by

$$w_{k,j} := \begin{cases} |\det S_j|^{\frac{1}{p_2}-1}, & \text{if } q_k < 1, \\ |\det S_j|^{\frac{1}{p_2}-\frac{1}{q_k}}, & \text{if } q_k \geq 1 \end{cases} \quad (5.4)$$

and

$$v_{k,i} := \begin{cases} |\det T_i|^{\frac{1}{p_1}-\frac{1}{q_k}} \cdot [\sup_{j \in J^{(k)}} \lambda(\overline{P_j} - \overline{Q_i})]^{\frac{1}{q_k}-1}, & \text{if } q_k < 1, \\ |\det T_i|^{\frac{1}{p_1}-\frac{1}{q_k}}, & \text{if } q_k \geq 1 \end{cases} \quad (5.5)$$

where we implicitly assume (for $k \in K$ with $q_k < 1$) that $\sup_{j \in J^{(k)}} \lambda(\overline{P_j} - \overline{Q_i}) < \infty$ for all $i \in I^{(k)}$.

If the maps

$$\eta_1 : X\left([\ell_w^{q_k}(J^{(k)})]_{k \in K}\right) \hookrightarrow Z, \quad (x_j)_{j \in J_{00}} \mapsto (x_j)_{j \in J} \text{ with } x_j = 0 \text{ for } j \in J \setminus J_{00} \quad (5.6)$$

$$\eta_2 : Y \rightarrow X\left([\ell_v^{q_k}(I^{(k)})]_{k \in K}\right), (x_i)_{i \in I} \mapsto (x_i)_{i \in L} \text{ with } L = \bigcup_{k \in K} I^{(k)} \subset I \quad (5.7)$$

are well-defined and bounded, the map

$$\iota : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z), f \mapsto \sum_{(i,j) \in I_0 \times J_0} \psi_j \varphi_i f,$$

is well-defined and bounded with $\|\iota\| \leq C \cdot \|\eta_1\| \cdot \|\eta_2\|$ for a constant

$$C = C\left(d, p_1, p_2, q^{(0)}, \mathcal{Q}, \mathcal{P}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, q^{(0)}}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}\right).$$

More precisely, we have (by definition)

$$\langle \iota f, g \rangle = \sum_{(i,j) \in I_0 \times J_0} \langle f, \varphi_i \psi_j g \rangle \quad \forall g \in C_c^\infty(\mathcal{O}'),$$

with absolute convergence of the series for all $g \in C_c^\infty(\mathcal{O}')$. ◀

Remark. A few remarks are in order:

- (1) The most common case is $J_0 = J$. In this case, note that since $(\psi_j)_{j \in J}$ is a (locally finite) partition of unity on \mathcal{O}' (cf. Lemma 2.4), we have $g = \sum_{j \in J} \psi_j g = \sum_{j \in J_0} \psi_j g$ for arbitrary $g \in C_c^\infty(\mathcal{O}')$, where only finitely many terms of the sum do not vanish identically. Hence,

$$\langle \iota f, g \rangle = \sum_{(i,j) \in I_0 \times J_0} \langle f, \varphi_i \psi_j g \rangle = \sum_{i \in I_0} \langle f, \varphi_i g \rangle.$$

Now, if furthermore $I_0 = I$, a similar argument shows for $g \in C_c^\infty(\mathcal{O} \cap \mathcal{O}')$ that $\langle \iota f, g \rangle = \langle f, g \rangle$.

In particular, if $I_0 = I$ and $J_0 = J$ and if furthermore $\mathcal{O} = \mathcal{O}'$, then $\iota f = f$ for all $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \leq \mathcal{D}'(\mathcal{O})$, so that ι is indeed an (injective) embedding.

- (2) In the present case and in future theorems, we always use the BAPUs Φ and Ψ not only to define the map ι , but also to compute the (quasi)-norms of the spaces $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$ and $\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$, respectively. Of course, other choices lead to equivalent norms (cf. Corollary 3.18), but in this case, the constant C from above would also depend on the BAPUs which are used to calculate the (quasi)-norms on the two decomposition spaces.

- (3) There is a slight variation of condition (5.3) which is sometimes useful: Assume that $I^{(k,0)} \subset I$ and $J^{(k)} \subset J$ satisfy

$$\mathcal{O} \cap \bigcup_{j \in J^{(k)}} P_j \subset \bigcup_{i \in I^{(k,0)}} Q_i.$$

Then the sets $J^{(k)}$ and $I^{(k)} := (I^{(k,0)})^*$ satisfy condition (5.3) with $I_0 := I$.

Indeed, if this was false, there would be some $\ell \in I \setminus I^{(k)}$ and some $h \in J^{(k)}$ with $Q_\ell \cap P_h \neq \emptyset$. Thus, there is some

$$\xi \in Q_\ell \cap P_h \subset \mathcal{O} \cap \bigcup_{j \in J^{(k)}} P_j \subset \bigcup_{i \in I^{(k,0)}} Q_i,$$

so that $\xi \in Q_i$ for some $i \in I^{(k,0)}$. But then $\xi \in Q_i \cap Q_\ell \neq \emptyset$ and hence $\ell \in i^* \subset (I^{(k,0)})^* = I^{(k)}$, in contradiction to $\ell \in I \setminus I^{(k)}$.

- (4) Note that there is no symmetry between \mathcal{Q}, \mathcal{P} in condition (5.3), i.e. the same condition is not “automatically” satisfied with \mathcal{Q}, \mathcal{P} interchanged. This is natural; as we will see in more detail in the proof, if we want to estimate $\|\iota f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)}$, we have to estimate the individual “pieces”

$$\begin{aligned} \|\mathcal{F}^{-1}(\psi_j \cdot \iota f)\|_{L^{p_2}} &= \left\| \mathcal{F}^{-1} \left(\psi_j \cdot \sum_{(i,\ell) \in I_0 \times J_0} \varphi_i \psi_\ell f \right) \right\|_{L^{p_2}} \\ &= \left\| \mathcal{F}^{-1} \left(\sum_{\ell \in J_0 \cap j^*} \psi_j \psi_\ell \sum_{i \in I_0} \varphi_i f \right) \right\|_{L^{p_2}} \\ &\lesssim \sum_{\ell \in J_0 \cap j^*} \left\| \mathcal{F}^{-1} \left(\psi_\ell \cdot \sum_{i \in I_0} \varphi_i f \right) \right\|_{L^{p_2}}. \end{aligned}$$

At this point, condition (5.3) is designed to ensure that we have $\psi_\ell \cdot \sum_{i \in I_0} \varphi_i f = \psi_\ell \cdot \sum_{i \in I^{(k)}} \varphi_i f$, as long as $\ell \in J^{(k)}$ for some fixed $k \in K$.

In other words, given $\ell \in J^{(k)}$, condition (5.3) allows us to identify those $i \in I$ such that $\varphi_i f$ (potentially) has an impact on $\psi_\ell \sum_{i \in I_0} \varphi_i f$. The point here is that we want to estimate ιf when localized using $\Psi = (\psi_j)_{j \in J}$, while we are given information about f localized using $\Phi = (\varphi_i)_{i \in I}$. Hence, we need information on how the localization using Ψ relates to Φ . This is exactly what condition (5.3) achieves.

Very briefly, there is no symmetry in (5.3) regarding \mathcal{Q}, \mathcal{P} , since there is no symmetry in the embedding $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$ regarding \mathcal{Q}, \mathcal{P} . ♦

Proof of Theorem 5.6. We divide the proof into 7 steps.

Step 1: Let $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$, define $c_i := \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^{p_1}}$ for $i \in I$ and note that $c := (c_i)_{i \in I} \in Y$ by definition of $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$. Since $\eta_2 : Y \rightarrow X \left(\left[\ell_v^{q_k^\vee}(I^{(k)}) \right]_{k \in K} \right)$ is well-defined, we see in particular that $\|(v_{k,i} \cdot c_i)_{i \in I^{(k)}}\|_{\ell^{q_k^\vee}}$ is finite for every $k \in K$. But because of $q_k^\vee \leq 2 < \infty$, this shows that there is a countable subset $I^{(k,0)} \subset I^{(k)}$ satisfying $\|\mathcal{F}^{-1}(\varphi_i f)\|_{L^{p_1}} = c_i = 0$, and thus $\varphi_i f \equiv 0$, for all $i \in I^{(k)} \setminus I^{(k,0)}$.

Now, an application of Lemma 5.3 shows (for $k \in K$ and $i \in I^{(k)}$, because of $q_k \geq p_1$) that

$$c_i^{(k)} := \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^{q_k}} \leq C_1 \cdot |\det T_i|^{\frac{1}{p_1} - \frac{1}{q_k}} \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^{p_1}} = C_1 \cdot |\det T_i|^{\frac{1}{p_1} - \frac{1}{q_k}} \cdot c_i$$

for some constant $C_1 = C_1(\mathcal{Q}, d, p_1)$.

Hence, we get for

$$u_{k,i} := |\det T_i|^{\frac{1}{q_k} - \frac{1}{p_1}} \cdot v_{k,i} = \begin{cases} \sup_{j \in J^{(k)}} [\lambda(\overline{P_j} - \overline{Q_i})]^{\frac{1}{q_k} - 1}, & \text{if } q_k < 1, \\ 1, & \text{if } q_k \geq 1 \end{cases}$$

that

$$\|(c_i^{(k)})_{i \in I^{(k)}}\|_{\ell_u^{q_k^\vee}} \leq C_1 \cdot \|(v_{k,i} \cdot c_i)_{i \in I^{(k)}}\|_{\ell^{q_k^\vee}} < \infty$$

for each $k \in K$.

Step 2: Fix $k \in K$. For $j \in J^{(k)}$ and $i \in I^{(k)}$, define $\theta_{k,j,i} := \|\mathcal{F}^{-1}(\psi_j \varphi_i f)\|_{L^{q_k}}$ and

$$\begin{aligned} d_{k,j,i} &:= w_{k,j} \cdot |\det S_j|^{\frac{1}{q_k} - \frac{1}{p_2}} \cdot \theta_{k,j,i} \\ &= w_{k,j} \cdot |\det S_j|^{\frac{1}{q_k} - \frac{1}{p_2}} \cdot \|\mathcal{F}^{-1}(\psi_j \varphi_i f)\|_{L^{q_k}} \\ &= \begin{cases} |\det S_j|^{\frac{1}{q_k} - 1} \cdot \|\mathcal{F}^{-1}(\psi_j \varphi_i f)\|_{L^{q_k}} = |\det S_j|^{\frac{1}{q_k} - 1} \theta_{k,j,i}, & \text{if } q_k < 1, \\ \|\mathcal{F}^{-1}(\psi_j \varphi_i f)\|_{L^{q_k}} = \theta_{k,j,i}, & \text{if } q_k \geq 1. \end{cases} \end{aligned} \quad (5.8)$$

As suggested by the definition of $d_{k,j,i}$ and of the weight $u = (u_{k,i})_{k \in K, i \in I^{(k)}}$ from above, we now distinguish two cases.

Case 1. We have $q_k \in (0, 1)$. Note that this implies $q_k^\Delta = \infty$. Now, we note $\text{supp } \varphi_i \subset \overline{Q_i}$ and $\text{supp } \psi_j \subset \overline{P_j}$ and use Theorem 3.4 to conclude for each $j \in J^{(k)}$ and $i \in I^{(k)}$ that

$$\begin{aligned} d_{k,j,i} &= |\det S_j|^{\frac{1}{q_k} - 1} \cdot \|\mathcal{F}^{-1}(\psi_j \varphi_i f)\|_{L^{q_k}} \\ &\leq [\lambda(\overline{P_j} - \overline{Q_i})]^{\frac{1}{q_k} - 1} \cdot |\det S_j|^{\frac{1}{q_k} - 1} \cdot \|\mathcal{F}^{-1} \psi_j\|_{L^{q_k}} \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^{q_k}} \\ &\leq [\lambda(\overline{P_j} - \overline{Q_i})]^{\frac{1}{q_k} - 1} \cdot C_{\mathcal{P}, \Psi, q_k} \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^{q_k}} \\ &\leq C_2 \cdot u_{k,i} \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^{q_k}} < \infty. \end{aligned}$$

Here, the last step used the definition of $u = (u_{k,i})_{k \in K, i \in I^{(k)}}$ (and that $j \in J^{(k)}$). Furthermore, it was used that Ψ is an $L^{q^{(0)}}$ -BAPU for \mathcal{P} and that $q_k \geq q^{(0)}$, so that Corollary 5.4 shows that Ψ is also an L^{q_k} -BAPU for \mathcal{P} , with $C_{\mathcal{P}, \Psi, q_k} \leq C_2 = C_2(d, \mathcal{P}, q^{(0)}, C_{\mathcal{P}, \Psi, q^{(0)}})$.

Because of $q_k^\Delta = \infty$ and since $j \in J^{(k)}$ was arbitrary, we finally get

$$\left\| (d_{k,j,i})_{j \in J^{(k)}} \right\|_{\ell^{q_k^\Delta}} \leq C_2 \cdot u_{k,i} \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^{q_k}} < \infty$$

for all $i \in I^{(k)}$.

Case 2. We have $q_k \in [1, \infty]$. In this case, Lemma 5.2 shows that for arbitrary $p \in [1, \infty]$, the map

$$\Gamma_p^{(0)} : L^p(\mathbb{R}^d) \rightarrow \ell^{p^\Delta}(J; L^p(\mathbb{R}^d)), g \mapsto (\mathcal{F}^{-1}(\psi_j \cdot \widehat{g}))_{j \in J}$$

is a well-defined, bounded, linear operator with $\left\| \Gamma_p^{(0)} \right\| \leq N_{\mathcal{P}} \cdot C_{\mathcal{P}, \Psi, 1} =: C_3$.

We now use this with $p = q_k \in [1, \infty]$ for $g = \mathcal{F}^{-1}(\varphi_i f) \in L^{q_k}(\mathbb{R}^d)$, to conclude for each $i \in I^{(k)}$ that

$$\begin{aligned} \left\| (d_{k,j,i})_{j \in J^{(k)}} \right\|_{\ell^{q_k^\Delta}} &= \left\| (\|\mathcal{F}^{-1}(\psi_j \varphi_i f)\|_{L^{q_k}})_{j \in J^{(k)}} \right\|_{\ell^{q_k^\Delta}} \\ &\leq \left\| (\|\mathcal{F}^{-1}(\psi_j \varphi_i f)\|_{L^{q_k}})_{j \in J} \right\|_{\ell^{q_k^\Delta}} \\ &\leq C_3 \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^{q_k}} \\ &= C_3 \cdot u_{k,i} \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^{q_k}} < \infty. \end{aligned}$$

All in all, we have shown that

$$\left\| (d_{k,j,i})_{j \in J^{(k)}} \right\|_{\ell^{q_k^\Delta}} \leq C_4 \cdot u_{k,i} \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^{q_k}} < \infty \quad (5.9)$$

holds in both cases for all $i \in I^{(k)}$, for a suitable constant $C_4 = C_4(d, \mathcal{P}, q^{(0)}, C_{\mathcal{P}, \Psi, 1}, C_{\mathcal{P}, \Psi, q^{(0)}})$.

Step 3: Now, take the $\ell^{q_k^\nabla}(I^{(k)})$ -norm of estimate (5.9) to derive

$$\begin{aligned} \left\| \left(\left\| (d_{k,j,i})_{j \in J^{(k)}} \right\|_{\ell^{q_k^\Delta}} \right)_{i \in I^{(k)}} \right\|_{\ell^{q_k^\nabla}} &\leq C_4 \cdot \left\| (u_{k,i} \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^{q_k}})_{i \in I^{(k)}} \right\|_{\ell^{q_k^\nabla}} \\ &= C_4 \cdot \left\| (c_i^{(k)})_{i \in I^{(k)}} \right\|_{\ell_u^{q_k^\nabla}} \\ (\text{Step 1}) &\leq C_1 C_4 \cdot \left\| (c_i)_{i \in I^{(k)}} \right\|_{\ell_v^{q_k^\nabla}} < \infty. \end{aligned}$$

Note that $q_k^\nabla = \min \{q_k, q'_k\} \leq \max \{q_k, q'_k\} = q_k^\triangle$. Thus, an application of Lemma 5.5 finally implies

$$\begin{aligned} \left\| \left(\| (d_{k,j,i})_{i \in I^{(k)}} \|_{\ell^{q_k^\nabla}} \right)_{j \in J^{(k)}} \right\|_{\ell^{q_k^\triangle}} &\leq \left\| \left(\| (d_{k,j,i})_{j \in J^{(k)}} \|_{\ell^{q_k^\triangle}} \right)_{i \in I^{(k)}} \right\|_{\ell^{q_k^\nabla}} \\ &\leq C_1 C_4 \cdot \| (c_i)_{i \in I^{(k)}} \|_{\ell^{q_k^\nabla}} < \infty \end{aligned} \quad (5.10)$$

for all $k \in K$.

Step 4: The result from the previous step implies in particular for each $k \in K$ and all $j \in J^{(k)}$ that

$$\begin{aligned} \| (\theta_{k,j,i})_{i \in I^{(k)}} \|_{\ell^{q_k^\nabla}} &= \left\| \left(\| \mathcal{F}^{-1}(\psi_j \varphi_i f) \|_{L^{q_k}} \right)_{i \in I^{(k)}} \right\|_{\ell^{q_k^\nabla}} \\ &= \begin{cases} |\det S_j|^{1-\frac{1}{q_k}} \cdot \| (d_{k,j,i})_{i \in I^{(k)}} \|_{\ell^{q_k^\nabla}} =: C_{k,j} < \infty, & \text{if } q_k < 1, \\ \| (d_{k,j,i})_{i \in I^{(k)}} \|_{\ell^{q_k^\nabla}} =: C_{k,j} < \infty, & \text{if } q_k \geq 1. \end{cases} \end{aligned} \quad (5.11)$$

We will derive from this that the mapping

$$f_{k,j} : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}, g \mapsto \sum_{i \in I^{(k)}} \langle \varphi_i f, \psi_j g \rangle \quad (5.12)$$

is a well-defined, tempered distribution for all $k \in K$ and each $j \in J^{(k)}$, with absolute convergence of the defining series. Since this is only a qualitative statement, we will suppress some unimportant constants in this step.

Fix $k \in K$ and $j \in J^{(k)}$. To prove the absolute convergence, we first note $\text{supp}(\varphi_i \psi_j f) \subset \overline{Q_i}$ and invoke Lemma 5.1 to derive for arbitrary finite subsets $F^{(k)} \subset I^{(k)}$ and

$$f_{F^{(k)}} := \sum_{i \in F^{(k)}} \varphi_i \psi_j f \in \mathcal{S}'(\mathbb{R}^d)$$

the estimate

$$\| \mathcal{F}^{-1} f_{F^{(k)}} \|_{L^{q_k}} \leq C_5 \cdot \left\| \left(\| \mathcal{F}^{-1}(\varphi_i \psi_j f) \|_{L^{q_k}} \right)_{i \in F^{(k)}} \right\|_{\ell^{q_k^\nabla}} = C_5 \cdot \| (\theta_{k,j,i})_{i \in F^{(k)}} \|_{\ell^{q_k^\nabla}} \quad (5.13)$$

for some constant $C_5 = C_5(N_{\mathcal{Q}}, C_{\mathcal{Q}, \Phi, 1})$. But Corollary 5.4 yields $C_{\mathcal{Q}, \Phi, 1} = C_{\mathcal{Q}, \Phi, \infty} \lesssim_{\mathcal{Q}, p_1, d} C_{\mathcal{Q}, \Phi, p_1}$, so that we can choose $C_5 = C_5(\mathcal{Q}, p_1, d, C_{\mathcal{Q}, \Phi, p_1})$.

Now, we claim that there is some $r_k \in [1, \infty]$ such that we have

$$\| \mathcal{F}^{-1} f_{F^{(k)}} \|_{L^{r_k}} \lesssim_{\mathcal{Q}, \Phi, k, j} \| (\theta_{k,j,i})_{i \in F^{(k)}} \|_{\ell^{q_k^\nabla}} \quad (5.14)$$

for all finite subsets $F^{(k)} \subset I^{(k)}$. Here, the implied constant *is allowed* to depend on $k \in K$ and on $j \in J^{(k)}$, but not on $F^{(k)} \subset I^{(k)}$.

In case of $q_k \in [1, \infty]$, we can simply take $r_k = q_k$, so that we can assume $q_k \in (0, 1)$. In this case, note that we have $\text{supp} f_{F^{(k)}} \subset \overline{P_j}$ (since we multiply with ψ_j), so that Corollary 3.3 implies

$$\begin{aligned} \| \mathcal{F}^{-1} f_{F^{(k)}} \|_{L^\infty} &\leq [\lambda(\overline{P_j})]^{1/q_k} \cdot \| \mathcal{F}^{-1} f_{F^{(k)}} \|_{L^{q_k}} \\ (\text{using eq. (5.13)}) &\lesssim_{\mathcal{Q}, \Phi, k, j} \| (\theta_{k,j,i})_{i \in F^{(k)}} \|_{\ell^{q_k^\nabla}}. \end{aligned}$$

Hence, we can choose $r_k = \infty$ in case of $q_k \in (0, 1)$.

Now, we can prove absolute convergence of the series defining $\langle f_{k,j}, g \rangle$ for $g \in \mathcal{S}(\mathbb{R}^d)$. Indeed, because of $q_k^\nabla \leq 2 < \infty$ and since $\| (\theta_{k,j,i})_{i \in I^{(k)}} \|_{\ell^{q_k^\nabla}} < \infty$, there is for arbitrary $\varepsilon > 0$ some finite subset $F^{(k, \varepsilon)} \subset I^{(k)}$ (potentially depending on $j \in J^{(k)}$) satisfying

$$\| (\theta_{k,j,i})_{i \in F^{(k)}} \|_{\ell^{q_k^\nabla}} \leq \varepsilon \quad \forall \text{ finite subsets } F^{(k)} \subset I^{(k)} \setminus F^{(k, \varepsilon)}.$$

But in view of estimate (5.14), this implies

$$\begin{aligned}
 \left| \sum_{i \in F^{(k)}} \langle \varphi_i f, \psi_j g \rangle \right| &= |\langle f_{F^{(k)}}, g \rangle| \\
 &= |\langle \mathcal{F}^{-1} f_{F^{(k)}}, \widehat{g} \rangle| \\
 &\leq \|\mathcal{F}^{-1} f_{F^{(k)}}\|_{L^{r_k}} \cdot \|\widehat{g}\|_{L^{r'_k}} \\
 &\lesssim_{\mathcal{Q}, \Phi, k, j} \|(\theta_{k,j,i})_{i \in F^{(k)}}\|_{\ell^{q_k^\nabla}} \cdot \|\widehat{g}\|_{L^{r'_k}} \\
 &\leq \varepsilon \cdot \|\widehat{g}\|_{L^{r'_k}}
 \end{aligned}$$

for every finite subset $F^{(k)} \subset I^{(k)} \setminus F^{(k,\varepsilon)}$. Since $\varepsilon > 0$ was arbitrary, this easily implies absolute convergence of the series $\sum_{i \in I} \langle \varphi_i f, \psi_j g \rangle = \langle f_{k,j}, g \rangle$ for each $g \in \mathcal{S}(\mathbb{R}^d)$.

Furthermore, as seen in Step 1, there is a countable subset $I^{(k,0)} \subset I^{(k)}$ satisfying $\varphi_i f \equiv 0$ for all $i \in I^{(k)} \setminus I^{(k,0)}$. In particular, this implies $\langle \varphi_i f, \psi_j g \rangle = 0$ for all $i \in I^{(k)} \setminus I^{(k,0)}$. Thus, if $(I^{(k,N)})_{N \in \mathbb{N}}$ is a nondecreasing family of finite sets with $I^{(k,0)} = \bigcup_{N \in \mathbb{N}} I^{(k,N)}$, then

$$\begin{aligned}
 |\langle f_{k,j}, g \rangle| &= \lim_{N \rightarrow \infty} \left| \sum_{i \in I^{(k,N)}} \langle \varphi_i f, \psi_j g \rangle \right| \\
 &= \lim_{N \rightarrow \infty} |\langle f_{I^{(k,N)}}, g \rangle| \\
 &= \lim_{N \rightarrow \infty} |\langle \mathcal{F}^{-1} f_{I^{(k,N)}}, \widehat{g} \rangle| \\
 &\leq \lim_{N \rightarrow \infty} \|\mathcal{F}^{-1} f_{I^{(k,N)}}\|_{L^{r_k}} \cdot \|\widehat{g}\|_{L^{r'_k}} \\
 (\text{equation (5.14)}) &\lesssim_{\mathcal{Q}, \Phi, k, j} \lim_{N \rightarrow \infty} \|(\theta_{k,j,i})_{i \in I^{(k,N)}}\|_{\ell^{q_k^\nabla}} \cdot \|\widehat{g}\|_{L^{r'_k}} \\
 &\leq \|(\theta_{k,j,i})_{i \in I^{(k)}}\|_{\ell^{q_k^\nabla}} \cdot \|\widehat{g}\|_{L^{r'_k}}.
 \end{aligned}$$

Because of $\|(\theta_{k,j,i})_{i \in I^{(k)}}\|_{\ell^{q_k^\nabla}} < \infty$ (cf. equation (5.11)) and since $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ and the inclusion $\mathcal{S}(\mathbb{R}^d) \hookrightarrow L^{r'_k}(\mathbb{R}^d)$ are continuous, we conclude $f_{k,j} \in \mathcal{S}'(\mathbb{R}^d)$, as claimed.

Step 5: Here, we continue the previous step by showing

$$\|\mathcal{F}^{-1} f_{k,j}\|_{L^{q_k}} \leq C_5 \cdot \|(\theta_{k,j,i})_{i \in I^{(k)}}\|_{\ell^{q_k^\nabla}} < \infty \quad (5.15)$$

for all $k \in K$ and $j \in J^{(k)}$. To see this, note that (since we multiply by ψ_j , cf. equation (5.12)), the tempered distribution $f_{k,j}$ has compact support $\text{supp } f_{k,j} \subset \overline{P_j}$. Thus, the Paley-Wiener theorem (cf. [16, Theorem 7.23]) implies that $\mathcal{F}^{-1} f_{k,j}$ is given by (integration against) the smooth function

$$\begin{aligned}
 (\mathcal{F}^{-1} f_{k,j})(x) &= \langle f_{k,j}, e^{2\pi i \langle x, \cdot \rangle} \rangle = \sum_{i \in I^{(k)}} \langle \varphi_i f, \psi_j e^{2\pi i \langle x, \cdot \rangle} \rangle \\
 (\varphi_i f &= 0 \text{ for } i \in I^{(k)} \setminus I^{(k,0)}) = \sum_{i \in I^{(k,0)}} \langle \varphi_i f, \psi_j e^{2\pi i \langle x, \cdot \rangle} \rangle \\
 &= \lim_{N \rightarrow \infty} \sum_{i \in I^{(k,N)}} \langle \varphi_i f, \psi_j e^{2\pi i \langle x, \cdot \rangle} \rangle \\
 &= \lim_{N \rightarrow \infty} (\mathcal{F}^{-1} f_{I^{(k,N)}})(x),
 \end{aligned}$$

so that Fatou's lemma yields

$$\begin{aligned}
 \|\mathcal{F}^{-1} f_{k,j}\|_{L^{q_k}} &\leq \liminf_{N \rightarrow \infty} \|\mathcal{F}^{-1} f_{I^{(k,N)}}\|_{L^{q_k}} \\
 (\text{equation (5.13)}) &\leq C_5 \cdot \liminf_{N \rightarrow \infty} \|(\theta_{k,j,i})_{i \in I^{(k,N)}}\|_{\ell^{q_k^\nabla}} \\
 &\leq C_5 \cdot \|(\theta_{k,j,i})_{i \in I^{(k)}}\|_{\ell^{q_k^\nabla}} < \infty.
 \end{aligned}$$

Finiteness of the last term was shown in equation (5.11) above.

Step 6: Now, we show that $\iota f \in \mathcal{D}'(\mathcal{O}')$ is a well-defined distribution, with absolute convergence of the defining series. To this end, let $k \in K$ be arbitrary and fix $j_0 \in J^{(k)}$.

Now, for $\ell \in I_0 \setminus I^{(k)}$, we have by assumption (cf. equation (5.3)) that

$$Q_\ell \cap P_{j_0} \subset \left(\bigcup_{i \in I_0 \setminus I^{(k)}} Q_i \right) \cap \left(\bigcup_{j \in J^{(k)}} P_j \right) = \emptyset$$

and thus $\varphi_\ell \cdot \psi_{j_0} \equiv 0$ for all $\ell \in I_0 \setminus I^{(k)}$.

Thus, for arbitrary $g \in \mathcal{S}(\mathbb{R}^d)$, we have $\langle \varphi_i f, \psi_{j_0} g \rangle = 0$ for all $i \in I_0 \setminus I^{(k)}$, so that we get (simply by dropping vanishing terms)

$$\left\langle \sum_{i \in I_0} \varphi_i \psi_{j_0} f, g \right\rangle = \sum_{i \in I_0} \langle \varphi_i f, \psi_{j_0} g \rangle = \sum_{i \in I^{(k)}} \langle \varphi_i f, \psi_{j_0} g \rangle = \langle f_{k,j_0}, g \rangle.$$

In particular, by what we proved above (in Step 4), the series $\sum_{i \in I_0} \langle \varphi_i f, \psi_{j_0} g \rangle$ converges absolutely and we have

$$\sum_{i \in I_0} \varphi_i \psi_{j_0} f = f_{k,j_0} \quad \forall k \in K \text{ and } j_0 \in J^{(k)}. \quad (5.16)$$

We call this the **localization identity**. As a crucial observation, note that for fixed $j_0 \in J_{00}$, the left-hand side of the localization identity (5.16) is *independent* of the choice $k \in K$ (satisfying $j_0 \in J^{(k)}$). Hence, we may (and will) in the following write $f_{j_0} := f_{k,j_0}$ for all $j_0 \in J_{00} = \bigcup_{k \in K} J^{(k)}$ and arbitrary $k \in K$ with $j_0 \in J^{(k)}$.

Now, let $M \subset \mathcal{O}'$ be an arbitrary compact set. Since Lemma 2.4 shows that $(\psi_j)_{j \in J}$ is a locally finite partition of unity on \mathcal{O}' , the set

$$J_M := \{j \in J_0 \mid M \cap \text{supp } \psi_j \neq \emptyset\}$$

is finite. For $g \in C_c^\infty(\mathcal{O}')$ with $\text{supp } g \subset M$, this implies $\psi_j g \equiv 0$ for $j \in J_0 \setminus J_M$. Next, note that because of $J_M \subset J_0 \subset J_{00} = \bigcup_{k \in K} J^{(k)}$, there is for each $j \in J_M$ some $k_j \in K$ with $j \in J^{(k_j)}$. Hence, simply by dropping vanishing terms, we see

$$\begin{aligned} \sum_{(i,j) \in I_0 \times J_0} |\langle f, \varphi_i \psi_j g \rangle| &= \sum_{(i,j) \in I_0 \times J_M} |\langle f, \varphi_i \psi_j g \rangle| \\ &= \sum_{j \in J_M} \sum_{i \in I_0} |\langle f, \varphi_i \psi_j g \rangle| < \infty, \end{aligned}$$

since J_M is finite and since each of the series $\sum_{i \in I_0} \langle f, \varphi_i \psi_j g \rangle = \langle f_{k_j,j}, g \rangle$ for $j \in J_M$ converges absolutely, as seen above.

Entirely the same calculation, but without the absolute value, shows

$$\begin{aligned} \langle \iota f, g \rangle &= \sum_{(i,j) \in I_0 \times J_0} \langle f, \varphi_i \psi_j g \rangle \\ &= \sum_{j \in J_M} \sum_{i \in I_0} \langle f, \varphi_i \psi_j g \rangle \\ &= \sum_{j \in J_M} \langle f_{k_j,j}, g \rangle \\ &= \left\langle \sum_{j \in J_M} f_{k_j,j}, g \right\rangle. \end{aligned}$$

Here, the right-hand side $\sum_{j \in J_M} f_{k_j,j}$ is a (tempered) distribution, since it is a finite sum of tempered distributions. In particular, we see that $g \mapsto \langle \iota f, g \rangle$ is continuous when restricted to

$$C_M^\infty(\mathcal{O}') = \{g \in C_c^\infty(\mathcal{O}') \mid \text{supp } g \subset M\},$$

for *every* compact subset $M \subset \mathcal{O}'$. But by [16, Theorem 6.6], this already implies $\iota f \in \mathcal{D}'(\mathcal{O}')$, as desired.

Step 7: It remains to show $\iota f \in \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$, together with an accompanying (quasi)-norm estimate. As we will see below, the sequence $\varrho = (\varrho_j)_{j \in J}$ given by

$$\varrho_j := \begin{cases} \|\mathcal{F}^{-1} f_j\|_{L^{p_2}}, & \text{if } j \in J_0, \\ 0, & \text{if } j \notin J_0 \end{cases}$$

will be useful to us. For now, we will derive a bound for $\varrho_j = \|\mathcal{F}^{-1} f_j\|_{L^{p_2}}$, for $j \in J^{(k)} \cap J_0$ (and arbitrary $k \in K$). Note that we have $f_j = f_{k,j}$.

Note that $\psi_j^* \psi_j = \psi_j$, since $\psi_j^* \equiv 1$ on P_j , cf. Lemma 2.4. Hence,

$$\langle f_{k,j}, \psi_j^* g \rangle = \sum_{i \in I^{(k)}} \langle \varphi_i f, \psi_j \psi_j^* g \rangle = \sum_{i \in I^{(k)}} \langle \varphi_i f, \psi_j g \rangle = \langle f_{k,j}, g \rangle \quad \forall g \in \mathcal{S}(\mathbb{R}^d),$$

i.e. $f_{k,j} = \psi_j^* \cdot f_{k,j}$. Hence, Lemma 5.3 (with $k = 1$ and $p_0 = q^{(0)}$) yields—because of $p_2 \geq q_k \geq q^{(0)}$ —some constant $C_6 = C_6(d, \mathcal{P}, q^{(0)})$ which satisfies

$$\begin{aligned} \varrho_j &= \|\mathcal{F}^{-1} f_{k,j}\|_{L^{p_2}} = \|\mathcal{F}^{-1}(\psi_j^* \cdot f_{k,j})\|_{L^{p_2}} \\ &\leq C_6 \cdot |\det S_j|^{\frac{1}{q_k} - \frac{1}{p_2}} \cdot \|\mathcal{F}^{-1}(\psi_j^* \cdot f_{k,j})\|_{L^{q_k}} \\ &= C_6 \cdot |\det S_j|^{\frac{1}{q_k} - \frac{1}{p_2}} \cdot \|\mathcal{F}^{-1} f_{k,j}\|_{L^{q_k}} \\ &\quad (\text{eq. (5.15)}) \leq C_5 C_6 \cdot |\det S_j|^{\frac{1}{q_k} - \frac{1}{p_2}} \cdot \|(\theta_{k,j,i})_{i \in I^{(k)}}\|_{\ell^{q_k^*}} < \infty. \end{aligned} \quad (5.17)$$

Now, let $j \in J$ be arbitrary. Recall that $\psi_j \psi_{j_0} \not\equiv 0$ implies $j_0 \in j^*$. Thus, using the definition of ιf and the localization identity (5.16), we see

$$\begin{aligned} \psi_j \cdot \iota f &= \psi_j \cdot \sum_{(i,j_0) \in I_0 \times J_0} \psi_{j_0} \varphi_i f \\ &= \sum_{j_0 \in J_0 \cap j^*} \left[\psi_j \cdot \sum_{i \in I_0} \psi_{j_0} \varphi_i f \right] \\ &= \sum_{j_0 \in J_0 \cap j^*} \psi_j f_{j_0}, \end{aligned}$$

where we used the notation f_{j_0} from Step 6. We recall that f_{j_0} satisfies $f_{j_0} = f_{k,j_0}$ for every $k \in K$ with $j_0 \in J^{(k)}$.

Now, we use the quasi-triangle inequality for L^{p_2} and the uniform bound $|j^*| \leq N_{\mathcal{P}}$ to obtain constants $C_7 = C_7(N_{\mathcal{P}}, p_2) > 0$ and $C_8 = C_8(C_{\mathcal{P}, \Psi, 1}, C_{\mathcal{P}, \Psi, p_2}, \mathcal{P}, d, p_2)$ with

$$\begin{aligned} \|\mathcal{F}^{-1}(\psi_j \cdot \iota f)\|_{L^{p_2}} &\leq C_7 \cdot \sum_{j_0 \in J_0 \cap j^*} \|\mathcal{F}^{-1}(\psi_j f_{j_0})\|_{L^{p_2}} \\ &\stackrel{(*)}{\leq} C_7 C_8 \cdot \sum_{j_0 \in J_0 \cap j^*} \|\mathcal{F}^{-1} f_{j_0}\|_{L^{p_2}} \\ &= C_7 C_8 \cdot \varrho_j^*, \end{aligned} \quad (5.18)$$

where $\varrho^* = (\varrho_j^*)_{j \in J}$ is the “clustered version” of the sequence ϱ , given by $\varrho_j^* = \sum_{\ell \in j^*} \varrho_\ell$, where the cluster j^* is taken with respect to \mathcal{P} . To justify the step marked with $(*)$, we distinguish two cases:

Case 1. $p_2 \in [1, \infty]$. In this case, we can simply use Young’s inequality and the uniform bound $\|\mathcal{F}^{-1} \psi_j\|_{L^1} \leq C_{\mathcal{P}, \Psi, 1}$ to conclude

$$\begin{aligned} \|\mathcal{F}^{-1}(\psi_j f_{j_0})\|_{L^{p_2}} &= \|\mathcal{F}^{-1} \psi_j * \mathcal{F}^{-1} f_{j_0}\|_{L^{p_2}} \\ &\leq \|\mathcal{F}^{-1} \psi_j\|_{L^1} \cdot \|\mathcal{F}^{-1} f_{j_0}\|_{L^{p_2}} \\ &\leq C_{\mathcal{P}, \Psi, 1} \cdot \|\mathcal{F}^{-1} f_{j_0}\|_{L^{p_2}}, \end{aligned}$$

as desired.

Case 2. $p_2 \in (0, 1)$. Here, by definition of an L^{p_2} -BAPU, $\|\mathcal{F}^{-1}\psi_j\|_{L^{p_2}} \leq C_{\mathcal{P}, \Psi, p_2} \cdot |\det S_j|^{1-\frac{1}{p_2}}$ for all $j \in J$, so that Corollary 3.14 implies because of $\text{supp } f_{j_0} \subset \overline{P_{j_0}}$ (note that we multiply with ψ_{j_0} in the definition of f_{j_0}) that

$$\begin{aligned} \|\mathcal{F}^{-1}(\psi_j f_{j_0})\|_{L^{p_2}} &\leq C(\mathcal{P}, d, p_2) \cdot |\det S_j|^{\frac{1}{p_2}-1} \cdot \|\mathcal{F}^{-1}\psi_j\|_{L^{p_2}} \|\mathcal{F}^{-1}f_{j_0}\|_{L^{p_2}} \\ &\leq C(\mathcal{P}, d, p_2) C_{\mathcal{P}, \Psi, p_2} \cdot \|\mathcal{F}^{-1}f_{j_0}\|_{L^{p_2}} \end{aligned}$$

for all $j_0 \in J_0 \cap j^*$.

Now, we multiply equation (5.17) by $w_{k,j}$ and recall the definition of the sequence $(d_{k,j,i})$ from equation (5.8), to derive

$$\begin{aligned} w_{k,j} \cdot \varrho_j &\leq C_5 C_6 \cdot |\det S_j|^{\frac{1}{q_k}-\frac{1}{p_2}} w_{k,j} \cdot \|(\theta_{k,j,i})_{i \in I(k)}\|_{\ell_k^{q_k^\vee}} \\ &= C_5 C_6 \cdot \|(d_{k,j,i})_{i \in I(k)}\|_{\ell_k^{q_k^\vee}}. \end{aligned} \quad (5.19)$$

Finally, we use the \mathcal{P} -regularity of Z , the embeddings η_1, η_2 and the solidity of Z and X to conclude

$$\begin{aligned} \|\iota f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)} &= \left\| \left(\|\mathcal{F}^{-1}(\psi_j \cdot \iota f)\|_{L^{p_2}} \right)_{j \in J} \right\|_Z \\ &\quad (\text{eq. (5.18)}) \leq C_7 C_8 \cdot \left\| (\varrho_j^*)_{j \in J} \right\|_Z \\ &\quad (Z \text{ is } \mathcal{P}\text{-regular}) \leq C_7 C_8 \cdot \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z} \cdot \|\varrho\|_Z \\ &\quad (\|\eta_1\| < \infty, \varrho_j = 0 \forall j \in J \setminus J_0) \leq C_7 C_8 \cdot \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z} \|\eta_1\| \cdot \left\| \left(\left\| (\varrho_j)_{j \in J(k)} \right\|_{\ell_w^{q_k^\Delta}} \right)_{k \in K} \right\|_X \\ &\quad = C_7 C_8 \cdot \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z} \|\eta_1\| \cdot \left\| \left(\left\| (w_{k,j} \cdot \varrho_j)_{j \in J(k)} \right\|_{\ell_k^{q_k^\Delta}} \right)_{k \in K} \right\|_X \\ &\quad (\text{eq. (5.19)}) \leq C_5 C_6 C_7 C_8 \cdot \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z} \|\eta_1\| \cdot \left\| \left(\left\| (d_{k,j,i})_{i \in I(k)} \right\|_{\ell_k^{q_k^\vee}} \right)_{j \in J(k)} \right\|_{\ell_k^{q_k^\Delta}} \right\|_{k \in K} \Big\|_X \\ &\quad (\text{eq. (5.10)}) \leq C_1 C_4 C_5 C_6 C_7 C_8 \cdot \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z} \|\eta_1\| \cdot \left\| \left((c_i)_{i \in I(k)} \right)_{k \in K} \right\|_{\ell_0^{q_k^\vee}} \Big\|_X \\ &\quad (\eta_2 \text{ bounded}) \leq C_1 C_4 C_5 C_6 C_7 C_8 \cdot \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z} \|\eta_1\| \cdot \|\eta_2\| \cdot \|(c_i)_{i \in I}\|_Y \\ &\quad = C_1 C_4 C_5 C_6 C_7 C_8 \cdot \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z} \|\eta_1\| \cdot \|\eta_2\| \cdot \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)} < \infty. \end{aligned}$$

Finally, Corollary 5.4 yields $C_{\mathcal{P}, \Psi, 1} = C_{\mathcal{P}, \Psi, \infty} \lesssim_{\mathcal{P}, q^{(0)}, d} C_{\mathcal{P}, \Psi, q^{(0)}}$ and $C_{\mathcal{P}, \Psi, p_2} \lesssim_{\mathcal{P}, q^{(0)}, d} C_{\mathcal{P}, \Psi, q^{(0)}}$, since $q^{(0)} \leq q_k \leq p_2 \leq \infty$ for arbitrary $k \in K$, so that the constant C can be chosen as stated. \square

Of course, verifying the assumptions of Theorem 5.6 is not easy in general. Thus, we formulate two important special cases as corollaries. Our first result handles embeddings of a decomposition space w.r.t. a “fine” covering into one w.r.t. a “coarse” covering. The present result is similar to [23, Theorem 5.1.8] from my PhD thesis, but slightly more general.

Corollary 5.7. *Let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q_i' + b_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J} = (S_j P_j' + c_j)_{j \in J}$ be semi-structured admissible coverings of the open sets $\mathcal{O} \subset \mathbb{R}^d$ and $\mathcal{O}' \subset \mathbb{R}^d$, respectively. Let $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ be two solid sequence spaces which are \mathcal{Q} -regular and \mathcal{P} -regular, respectively.*

Let $p_1, p_2 \in (0, \infty]$ with $p_1 \leq p_2$ and assume that \mathcal{Q} admits an L^{p_1} -BAPU $\Phi = (\varphi_i)_{i \in I}$ and that \mathcal{P} admits an L^{p_2} -BAPU $\Psi = (\psi_j)_{j \in J}$.

Let $I_0 \subset I$ be arbitrary with $\mathcal{Q}_i \subset \mathcal{O}'$ for all $i \in I_0$. In case of $p_2 < 1$, assume additionally that the “restricted” family $\mathcal{Q}_{I_0} = (Q_i)_{i \in I_0}$ is almost subordinate to \mathcal{P} .

Finally, assume that the embedding⁷

$$\eta : Y|_{I_0} \hookrightarrow Z \left(\left[\ell^{p_2^\vee} \right]_{|\det T_i|^{p_1^{-1}-p_2^{-1}}} (I_0 \cap I_j) \right)_{j \in J}$$

⁷Recall the definition $Y|_{I_0} = \{c = (c_i)_{i \in I_0} \in \mathbb{C}^{I_0} \mid \tilde{c} \in Y\}$, where $\tilde{c} = (c_i)_{i \in I}$, with $c_i = 0$ for $i \in I \setminus I_0$, cf. equation (3.12).

is well-defined and bounded, with $I_j = \{i \in I \mid Q_i \cap P_j \neq \emptyset\}$.

Then, the map

$$\iota : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z), f \mapsto \sum_{i \in I_0} \varphi_i f$$

is well-defined and bounded with absolute convergence of the defining series and with $\|\iota\| \leq C \cdot \|\eta\|$ for some constant

$$C = C(d, p_1, p_2, k(\mathcal{Q}_{I_0}, \mathcal{P}), \mathcal{Q}, \mathcal{P}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}),$$

where the dependence on $k(\mathcal{Q}_{I_0}, \mathcal{P})$ can be dropped for $p_2 \in [1, \infty]$.

Finally, if $I_0 = I$, then the distribution $\iota f \in \mathcal{D}'(\mathcal{O}')$ is an extension of f to $C_c^\infty(\mathcal{O}') \supset C_c^\infty(\mathcal{O})$, for arbitrary $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \leq \mathcal{D}'(\mathcal{O})$. In particular, this implies that ι is injective. \blacktriangleleft

Proof. We want to apply Theorem 5.6 with I_0 as in the statement of the present corollary and the following additional choices:

$$K := J, \quad J^{(k)} := \{k\}, \quad I^{(k)} := I_k \cap I_0 = \{i \in I_0 \mid Q_i \cap P_k \neq \emptyset\} \quad \text{and} \quad q_k := p_2$$

for all $k \in K = J$. With these choices, it is clear that condition (5.3) is satisfied. Furthermore, $J_{00} = \bigcup_{k \in K} J^{(k)} = \bigcup_{k \in J} \{k\} = J$, so that we can choose $J_0 := J \subset J_{00}$. Finally, $q^{(0)} = \inf_{k \in K} q_k = p_2$.

Next, we estimate the weight v from equation (5.5). To this end, let $k \in K$ and $i \in I^{(k)}$ be arbitrary. For $p_2 = q_k \geq 1$, we simply have

$$v_{k,i} = |\det T_i|^{\frac{1}{p_1} - \frac{1}{q_k}} = |\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}}.$$

In case of $p_2 = q_k < 1$, our prerequisites include the assumption that \mathcal{Q}_{I_0} is almost subordinate to \mathcal{P} . Hence, set $N := k(\mathcal{Q}_{I_0}, \mathcal{P})$. Now, since $J^{(k)} = \{k\}$, we have

$$\sup_{j \in J^{(k)}} \lambda(\overline{P_j} - \overline{Q_i}) = \lambda(\overline{P_k} - \overline{Q_i}).$$

Furthermore, for every $i \in I^{(k)} = I_0 \cap I_k$, there is some $j_i \in J$ with

$$\emptyset \neq Q_i \cap P_k \subset P_{j_i}^{N*} \cap P_k$$

and thus $k \in j_i^{(N+1)*}$, i.e. $j_i \in k^{(N+1)*}$, so that we get $Q_i \subset P_{j_i}^{N*} \subset P_k^{(2N+1)*}$. Hence, Corollary 2.8 yields a constant $C_1 = C_1(\mathcal{P}, N, d) \geq 1$ satisfying

$$\begin{aligned} \sup_{j \in J^{(k)}} \lambda(\overline{P_j} - \overline{Q_i}) &= \lambda(\overline{P_k} - \overline{Q_i}) \\ &\leq \lambda(\overline{P_k^{(2N+1)*}} - \overline{P_k^{(2N+1)*}}) \\ &\leq C_1 \cdot |\det S_k|. \end{aligned}$$

All in all, for $C_2 := C_1^{\frac{1}{p_2} - 1} \geq 1$ (in case of $p_2 < 1$), we have shown

$$v_{k,i} \leq \begin{cases} C_2 \cdot |\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot |\det S_k|^{\frac{1}{p_2} - 1}, & \text{if } p_2 < 1, \\ |\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}}, & \text{if } p_2 \geq 1. \end{cases}$$

Now, our final choice for the application of Theorem 5.6 is to set $X := Z_{1/w} \leq \mathbb{C}^J = \mathbb{C}^K$, where the weight w is chosen as in equation (5.4), i.e.

$$w_j := w_{j,j} = w_{k,j} := \begin{cases} |\det S_j|^{\frac{1}{p_2} - 1}, & \text{if } p_2 < 1, \\ |\det S_j|^{\frac{1}{p_2} - \frac{1}{q_k}} = |\det S_j|^{\frac{1}{p_2} - \frac{1}{p_2}} = 1, & \text{if } p_2 \geq 1 \end{cases}$$

for $k \in K = J$ and $j \in J^{(k)} = \{k\}$, i.e. $j = k$. Using this choice and recalling $J^{(k)} = \{k\}$ for $k \in K = J$, we see

$$X \left(\left[\ell_w^{q_k^\Delta}(J^{(k)}) \right]_{k \in K} \right) = X_w = (Z_{1/w})_w = Z \hookrightarrow Z,$$

so that the map η_1 from condition (5.6) of Theorem 5.6 is well-defined and bounded with $\|\eta_1\| = 1$.

For boundedness of η_2 from condition (5.7) of Theorem 5.6, first note that every $i \in I_0$ satisfies $\emptyset \neq Q_i \subset \mathcal{O}'$ and hence $Q_i \cap P_{j_i} \neq \emptyset$ for some $j_i \in J$. In particular, there is some $k \in J = K$ satisfying $Q_i \cap P_k \neq \emptyset$, i.e. $i \in I^{(k)}$, so that we have $L = \bigcup_{k \in K} I^{(k)} = I_0$. Furthermore, observe

$$X \left(\left[\ell_v^{q_k} (I^{(k)}) \right]_{k \in K} \right) = Z \left(\left[\ell_{(v_{k,i}/w_k)_i}^{p_2} (I^{(k)}) \right]_{k \in K} \right) \text{ with identical (quasi)-norms,}$$

where

$$\begin{aligned} \frac{v_{k,i}}{w_k} &\leq \begin{cases} C_2 \cdot |\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot |\det S_k|^{\frac{1}{p_2} - 1} \cdot |\det S_k|^{1 - \frac{1}{p_2}}, & \text{if } p_2 < 1, \\ |\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}}, & \text{if } p_2 \geq 1 \end{cases} \\ &\leq C_3 \cdot |\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}}, \end{aligned} \quad (5.20)$$

with $C_1 := C_2$ if $p_2 < 1$ and $C_3 := 1$ if $p_2 \geq 1$.

Thus, our assumption (boundedness of η) shows that we have

$$\begin{aligned} Y|_{I_0} &\xrightarrow{\eta} Z \left(\left[\ell_{|\det T_i|^{p_1^{-1} - p_2^{-1}}}^{p_2} (I_0 \cap I_j) \right]_{j \in J} \right) \\ &\xrightarrow{\text{eq. (5.20) and } I^{(k)} = I_0 \cap I_k} Z \left(\left[\ell_{(v_{k,i}/w_k)_i}^{p_2} (I^{(k)}) \right]_{k \in K} \right) \\ &= X \left(\left[\ell_v^{q_k} (I^{(k)}) \right]_{k \in K} \right), \end{aligned}$$

which easily shows that the embedding η_2 from condition (5.7) is well-defined and bounded with $\|\eta_2\| \leq C_3 \cdot \|\eta\|$. Hence, an application of Theorem 5.6 (together with the first part of the ensuing remark) implies that ι is bounded with

$$\|\iota\| \leq C_4 \cdot \|\eta_1\| \cdot \|\eta_2\| \leq C_3 C_4 \cdot \|\eta\|$$

for some constant $C_4 = C_4(d, p_1, p_2, \mathcal{Q}, \mathcal{P}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z})$. This completes the proof of the first part of the corollary.

Finally, in case of $I_0 = I$, note that we have $Q_i \subset \mathcal{O}'$ for all $i \in I_0 = I$ and hence $\mathcal{O} = \bigcup_{i \in I} Q_i \subset \mathcal{O}'$, which implies $C_c^\infty(\mathcal{O}) \subset C_c^\infty(\mathcal{O}')$. Now, since $(\varphi_i)_{i \in I}$ is a (locally finite) partition of unity on \mathcal{O} (cf. Lemma 2.4), we have for every $g \in C_c^\infty(\mathcal{O}) \subset C_c^\infty(\mathcal{O}')$ that $g = \sum_{i \in I} \varphi_i g$, where only finitely many terms do not vanish identically. Hence,

$$\langle \iota f, g \rangle = \sum_{i \in I} \langle f, \varphi_i g \rangle = \left\langle f, \sum_{i \in I} \varphi_i g \right\rangle = \langle f, g \rangle,$$

so that ιf indeed extends f if $I_0 = I$. \square

If one wants to apply the preceding corollary, one faces two major challenges: First, one has to verify that \mathcal{Q} (or \mathcal{Q}_{I_0}) is almost subordinate to \mathcal{P} . Then, one has to verify boundedness of the embedding η ; in fact, it might also be desired to obtain a bound for $\|\eta\|$, since this yields a bound for $\|\iota\|$. Luckily, using the results from Section 4, one can greatly simplify this second part, at least if Y and Z are both weighted Lebesgue sequence spaces:

Corollary 5.8. *Let $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ be two admissible coverings of the open sets $\emptyset \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$, respectively. Furthermore, let $w = (w_i)_{i \in I}$ and $v = (v_j)_{j \in J}$ be \mathcal{Q} - and \mathcal{P} -moderate, respectively. Finally, let $q_1, q_2, r \in (0, \infty]$ and let $u = (u_i)_{i \in I}$ be a further weight on I (which is not necessarily \mathcal{Q} -moderate).*

If $I_0 \subset I$ is chosen such that $\mathcal{Q}_{I_0} = (Q_i)_{i \in I_0}$ is almost subordinate to \mathcal{P} and if $J_0 \subset J$ satisfies⁸ $I_0 \subset \bigcup_{j \in J_0} I_j$, then we have

$$\|\eta\| \asymp \left\| \left(v_j \cdot \left\| (u_i/w_i)_{i \in I_0 \cap I_j} \right\|_{\ell^{r \cdot (q_1/r)'}} \right)_{j \in J_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \quad (5.21)$$

⁸The typical choice will simply be $J_0 = J$. In this case, the conclusion $I_0 \subset \bigcup_{j \in J_0} I_j$ is always satisfied; indeed, for $i \in I_0$, we have $\emptyset \neq Q_i \subset \mathcal{O}'$, since \mathcal{Q}_{I_0} is almost subordinate to \mathcal{P} . Since \mathcal{P} covers \mathcal{O}' , this yields $Q_i \cap P_j \neq \emptyset$ for some $j \in J$ and hence $i \in I_j$.

for

$$\eta : \ell_w^{q_1}(I_0) \hookrightarrow \ell_v^{q_2} \left([\ell_u^r(I_0 \cap I_j)]_{j \in J_0} \right).$$

Precisely, this means that η is well-defined and bounded if and only if the right-hand side of equation (5.21) is finite. Furthermore, there is a constant $C \geq 1$ depending only on $q_1, q_2, r, N_{\mathcal{P}}, k(\mathcal{Q}_{I_0}, \mathcal{P}), C_{v, \mathcal{P}}$ which satisfies

$$C^{-1} \cdot M \leq \|\eta\| \leq C \cdot M \quad \text{for} \quad M := \left\| \left(v_j \cdot \left\| (u_i/w_i)_{i \in I_0 \cap I_j} \right\|_{\ell^{r \cdot (q_1/r)'} } \right)_{j \in J_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)' } }.$$

Finally, if

- $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ and $\mathcal{P} = (S_j P'_j + c_j)_{j \in J}$ are tight semi-structured coverings,
- \mathcal{Q}_{I_0} is relatively \mathcal{P} -moderate,
- there is some $s \in \mathbb{N}_0$ and some $C_0 > 0$ such that

$$\lambda(P_j) \leq C_0 \cdot \lambda \left(\bigcup_{i \in I_0 \cap I_j} Q_i^{s*} \right)$$

holds for all $j \in J_0$ with $1 \leq |I_0 \cap I_j| < \infty$,

- $u|_{I_0}$ and $w|_{I_0}$ are relatively \mathcal{P} -moderate and
- for each $j \in J^{(0)} := \{j \in J_0 \mid I_0 \cap I_j \neq \emptyset\}$, some $i_j \in I_0 \cap I_j$ is selected,

then

$$\left\| \left(v_j \cdot \left\| (u_i/w_i)_{i \in I_0 \cap I_j} \right\|_{\ell^{r \cdot (q_1/r)'} } \right)_{j \in J_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)' } } \asymp \left\| \left(\frac{v_j \cdot u_{i_j}}{w_{i_j}} \cdot [|\det S_j| / |\det T_{i_j}|]^{(\frac{1}{r} - \frac{1}{q_1})_+} \right)_{j \in J^{(0)}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)' } },$$

where the implied constant only depends on

$$d, s, r, q_1, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{Q}}, \varepsilon_{\mathcal{P}}, C_0, k(\mathcal{Q}_{I_0}, \mathcal{P}), C_{u|_{I_0}, \mathcal{Q}_{I_0}, \mathcal{P}}, C_{w|_{I_0}, \mathcal{Q}_{I_0}, \mathcal{P}}, C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}). \quad \blacktriangleleft$$

Remark 5.9. In particular, if we have $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$ in Corollary 5.7, then the embedding η from that corollary satisfies

$$\|\eta\| \asymp \left\| \left(v_j \cdot \left\| (|\det T_{i_j}|^{\frac{1}{p_1} - \frac{1}{p_2}} / w_{i_j})_{i \in I_0 \cap I_j} \right\|_{\ell^{p_2 \cdot (q_1/p_2)' } } \right)_{j \in J} \right\|_{\ell^{q_2 \cdot (q_1/q_2)' } },$$

where the implied constant only depends on $q_1, q_2, p_2, N_{\mathcal{P}}, k(\mathcal{Q}_{I_0}, \mathcal{P}), C_{v, \mathcal{P}}$.

Furthermore, if the additional assumptions from the second part of the corollary are satisfied, we have

$$\begin{aligned} \|\eta\| &\asymp \left\| \left(\frac{v_j}{w_{i_j}} \cdot |\det T_{i_j}|^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot [|\det S_j| / |\det T_{i_j}|]^{(\frac{1}{p_2} - \frac{1}{q_1})_+} \right)_{j \in J^{(0)}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)' } } \\ &= \left\| \left(\frac{v_j}{w_{i_j}} \cdot |\det T_{i_j}|^{\frac{1}{p_1} - (\frac{1}{p_2} - \frac{1}{q_1})_+ - \frac{1}{p_2}} \cdot |\det S_j|^{(\frac{1}{p_2} - \frac{1}{q_1})_+} \right)_{j \in J^{(0)}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)' } } \end{aligned}$$

where the implied constant only depends on

$$d, s, r, q_1, q_2, p_1, p_2, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{Q}}, \varepsilon_{\mathcal{P}}, C_0, k(\mathcal{Q}_{I_0}, \mathcal{P}), C_{w|_{I_0}, \mathcal{Q}_{I_0}, \mathcal{P}}, C_{v, \mathcal{P}}, C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}). \quad \blacklozenge$$

Proof. We begin with the first part of the corollary. For the proof, we want to use the estimate provided by parts (2) and (3) of Corollary 4.12 with $K = J_0$, $I = I_0$, $X = \ell_v^{q_2}(K) = \ell_v^{q_2}(J_0)$ and

$$I^{(k, \mathfrak{h})} = I^{(k)} := I_0 \cap I_k = \{i \in I_0 \mid Q_i \cap P_k \neq \emptyset\}$$

for $k \in K = J_0$. For this, we first have to verify the prerequisites of Corollary 4.12, which are just the prerequisites of Lemma 4.9.

To this end, first note that we have $I_0 = \bigcup_{k \in K} I^{(k)}$. Indeed, “ \supset ” is trivial; and “ \subset ” follows from our assumption $I_0 \subset \bigcup_{j \in J_0} I_j$

Now, define a relation \sim on $J_0 = K$ by $j \sim \ell : \iff I^{(j)} \cap I^{(\ell)} \neq \emptyset \iff I_0 \cap I_j \cap I_\ell \neq \emptyset$. For brevity, let us write $k := k(\mathcal{Q}_{I_0}, \mathcal{P}) \in \mathbb{N}_0$, which is well-defined, since \mathcal{Q}_{I_0} is almost subordinate to \mathcal{P} . Lemma 4.11 (with $n = 0$ and \mathcal{Q}_{I_0} instead of \mathcal{Q} and with \mathcal{P} instead of \mathcal{R}) implies⁹ that the classes $[j]$ of the given relation \sim satisfy

$$[j] \subset j^{(4k+5)*} \quad \text{as well as} \quad |[j]| \leq N_{\mathcal{P}}^{4k+5} =: N \quad \forall j \in J_0.$$

In particular, condition (4.5) of Lemma 4.9 is satisfied.

Furthermore, the generalized clustering map Θ from that Lemma satisfies

$$|(\Theta x)_j| \leq \sum_{\ell \in [j]} |x_\ell| \leq \sum_{\ell \in j^{(4k+5)*}} \mathbf{1}_{J_0}(\ell) \cdot |x_\ell| = (\Theta_{4k+5} [\mathbf{1}_{J_0} \cdot |x|])_j$$

for all $j \in J_0$ and $x = (x_j)_{j \in J_0} \in X$ (which we extended by 0 to all of J to obtain a sequence $(x_j)_{j \in J}$). Hence,

$$\begin{aligned} \|\Theta x\|_X &= \|\Theta x\|_{\ell_v^{q_2}(J_0)} \leq \|\Theta_{4k+5} [\mathbf{1}_{J_0} \cdot |x|]\|_{\ell_v^{q_2}(J_0)} \\ &\leq \|\Theta_{4k+5} [\mathbf{1}_{J_0} \cdot |x|]\|_{\ell_v^{q_2}(J)} \\ &\leq \|\Theta_{4k+5}\|_{\ell_v^{q_2}(J) \rightarrow \ell_v^{q_2}(J)} \cdot \|\mathbf{1}_{J_0} \cdot |x|\|_{\ell_v^{q_2}(J)} \\ &= \|\Theta_{4k+5}\|_{\ell_v^{q_2}(J) \rightarrow \ell_v^{q_2}(J)} \cdot \|x\|_{\ell_v^{q_2}(J_0)} \\ &\stackrel{(\text{Lemma 4.13})}{\leq} \left(C_{v,\mathcal{P}} \cdot N_{\mathcal{P}}^{1+\frac{1}{q_2}} \right)^{4k+5} \cdot \|x\|_X \end{aligned}$$

with Θ_n (for $n \in \mathbb{N}_0$) as in Lemma 3.9.

For the final assumption of Lemma 4.9 (regarding the “inner” weight $u = (u_{j,i})$), note that we have in our case $u_{j,i} = u_i$ for all $j \in J_0$ and $i \in I^{(j)} = I_0 \cap I_j$, so that $u_{j,i} \leq u_{\ell,i}$ holds for all $j, \ell \in J_0$ and $i \in I^{(j)} \cap I^{(\ell)}$, i.e. we can choose $C_u = 1$.

Now, we can finally apply Corollary 4.12 (with slightly permuted roles of the weights u, v, w and of the exponents of the weighted Lebesgue spaces) to get

$$\|\eta\| \asymp \left\| \left(v_j \cdot \left\| (u_i/w_i)_{i \in I_0 \cap I_j} \right\|_{\ell^{r \cdot (q_1/r)'} } \right)_{j \in J_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)' }},$$

where the implied constant only depends on $N_{\mathcal{P}}, k, q_1, q_2, r, C_{v,\mathcal{P}}$, since the triangle constant C_X of $X = \ell_v^{q_2}(J_0)$ only depends on q_2 .

Now, we consider the second part of the corollary. Note that the present assumptions include those of Lemma 2.17 (since \mathcal{P}_{J_0} -moderateness of \mathcal{Q}_{I_0} implies \mathcal{P} -moderateness of \mathcal{Q}_{I_0} ; precisely, we have $C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}_{J_0}) \leq C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P})$). Thus, that lemma yields

$$|I_0 \cap I_j| \asymp |\det S_j| / |\det T_{i_j}|$$

for each $j \in J_0$ with $I_0 \cap I_j \neq \emptyset$, i.e. for each $j \in J^{(0)}$. Here, the implied constant only depends on those quantities which are mentioned in the second part of the corollary.

But for $j \in J^{(0)}$ and $i \in I_0 \cap I_j$, we also have

$$\left(C_{w|_{I_0, \mathcal{Q}_{I_0}, \mathcal{P}}} \cdot C_{u|_{I_0, \mathcal{Q}_{I_0}, \mathcal{P}}} \right)^{-1} \cdot \frac{u_{i_j}}{w_{i_j}} \leq \frac{u_i}{w_i} \leq C_{w|_{I_0, \mathcal{Q}_{I_0}, \mathcal{P}}} \cdot C_{u|_{I_0, \mathcal{Q}_{I_0}, \mathcal{P}}} \cdot \frac{u_{i_j}}{w_{i_j}}.$$

All in all, this easily implies

$$\begin{aligned} \left\| (u_i/w_i)_{i \in I_0 \cap I_j} \right\|_{\ell^{r \cdot (q_1/r)' }} &\asymp \frac{u_{i_j}}{w_{i_j}} \cdot |I_0 \cap I_j|^{[r \cdot (q_1/r)']^{-1}} \\ &\asymp \frac{u_{i_j}}{w_{i_j}} \cdot [|\det S_j| / |\det T_{i_j}|]^{[r \cdot (q_1/r)']^{-1}} \\ &= \frac{u_{i_j}}{w_{i_j}} \cdot [|\det S_j| / |\det T_{i_j}|]^{(\frac{1}{r} - \frac{1}{q_1})_+^{-1}}, \end{aligned}$$

⁹Strictly speaking, we need to extend the relation \sim to all of J to apply Lemma 4.11. But this can be done simply by defining $I^{(k)} := \emptyset$ for $k \in J \setminus J_0$.

with implied constants as stated in the corollary. Here, the last step used $\frac{1}{a \cdot (b/a)^r} = \left(\frac{1}{a} - \frac{1}{b}\right)_+$, cf. equation (4.2).

Since the left-hand side of the desired estimate is unaffected if all terms $j \in J_0 \setminus J^{(0)}$ are discarded, this easily yields the claim. \square

In Corollary 5.7, we assumed \mathcal{Q} to be almost subordinate to \mathcal{P} . In our next result (which is similar to [23, Theorem 5.1.6] from my PhD thesis), we make the “reverse” assumption. To simplify the proof, we state the following technical lemma in advance.

Lemma 5.10. *Let $\mathcal{Q} = (Q_i)_{i \in I}$ be an admissible covering of a set \mathcal{O} and let $X, Y \leq \mathbb{C}^I$ be \mathcal{Q} -regular sequence spaces. Fix $n \in \mathbb{N}_0$ and assume that for each $i \in I$, some $q_i \in (0, \infty]$ and some subset $I_i \subset i^{n*}$ is given. Finally, assume that $q := \inf_{i \in I} q_i > 0$ and let $v = (v_i)_{i \in I}$ be an arbitrary (positive) weight.*

If $\iota : Y \hookrightarrow X_v$ is bounded, then so is $\theta : Y \hookrightarrow X([\ell_v^{q_i}(I_i)]_{i \in I})$, with

$$\|\theta\| \leq N_{\mathcal{Q}}^{n/q} \cdot \|\Gamma_{\mathcal{Q}}\|_{X \rightarrow X}^n \cdot \|\iota\|.$$

Proof. Let $x = (x_i)_{i \in I} \in Y$ be arbitrary. Let $i \in I$. Using the uniform bound $|I_i| \leq |i^{n*}| \leq N_{\mathcal{Q}}^n$ (cf. Lemma 2.9) and the norm-decreasing embedding $\ell^q \hookrightarrow \ell^{q_i}$, we get (for $q < \infty$) that

$$\begin{aligned} \|(x_\ell)_{\ell \in I_i}\|_{\ell_v^{q_i}} &\leq \|(x_\ell)_{\ell \in I_i}\|_{\ell_v^q} = \left[\sum_{\ell \in I_i} (v_\ell |x_\ell|)^q \right]^{1/q} \\ &\leq |I_i|^{1/q} \cdot \max_{\ell \in I_i} v_\ell |x_\ell| \\ &\leq N_{\mathcal{Q}}^{n/q} \cdot \sum_{\ell \in i^{n*}} |v_\ell x_\ell| \\ &= N_{\mathcal{Q}}^{n/q} \cdot (\Theta_n |v \cdot x|)_i, \end{aligned}$$

where Θ_n denotes the n -fold clustering map from Lemma 3.9 and where $|v \cdot x|_\ell = v_\ell \cdot |x_\ell|$. It is not hard to see that this estimate remains valid for $q = \infty$.

By solidity of X , we conclude

$$\begin{aligned} \|x\|_{X([\ell_v^{q_i}(I_i)]_{i \in I})} &\leq N_{\mathcal{Q}}^{n/q} \cdot \|\Theta_n |v \cdot x|\|_X \\ &\leq N_{\mathcal{Q}}^{n/q} \|\Theta_n\|_{X \rightarrow X} \cdot \|v \cdot x\|_X \\ &= N_{\mathcal{Q}}^{n/q} \|\Theta_n\|_{X \rightarrow X} \cdot \|x\|_{X_v} \\ &\leq N_{\mathcal{Q}}^{n/q} \|\Theta_n\|_{X \rightarrow X} \|\iota\| \cdot \|x\|_Y < \infty. \end{aligned}$$

Recalling the bound $\|\Theta_n\|_{X \rightarrow X} \leq \|\Gamma_{\mathcal{Q}}\|_{X \rightarrow X}^n$ from Lemma 3.9, we get the desired bound. \square

Corollary 5.11. *Let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J} = (S_j P'_j + c_j)_{j \in J}$ be semi-structured admissible coverings of the open sets $\mathcal{O} \subset \mathbb{R}^d$ and $\mathcal{O}' \subset \mathbb{R}^d$, respectively. Let $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ be two solid sequence spaces which are \mathcal{Q} -regular and \mathcal{P} -regular, respectively.*

Let $p_1, p_2 \in (0, \infty]$ with $p_1 \leq p_2$ and assume that \mathcal{Q} and \mathcal{P} admit L^{p_1} -BAPUs $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_j)_{j \in J}$, respectively.

Finally, let $J_0 \subset J$ and assume that the “restricted” family $\mathcal{P}_{J_0} = (P_j)_{j \in J_0}$ is almost subordinate to \mathcal{Q} and that

$$\eta : Y \left(\left[\ell_u^{\Delta p_1} (J_0 \cap J_i) \right]_{i \in I} \right) \hookrightarrow Z|_{J_0}$$

is well-defined and bounded, where $J_i = \{j \in J \mid P_j \cap Q_i \neq \emptyset\}$ and

$$u_{i,j} := \begin{cases} |\det S_j|^{\frac{1}{p_2}-1} \cdot |\det T_i|^{1-\frac{1}{p_1}}, & \text{if } p_1 < 1, \\ |\det S_j|^{\frac{1}{p_2}-\frac{1}{p_1}}, & \text{if } p_1 \geq 1. \end{cases} \quad (5.22)$$

Then, the map

$$\iota : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z), f \mapsto \sum_{j \in J_0} \psi_j f$$

is well-defined and bounded with absolute convergence of the defining series and with $\|\iota\| \leq C \cdot \|\eta\|$ for some constant

$$C = C(d, p_1, p_2, k(\mathcal{P}_{J_0}, \mathcal{Q}), \mathcal{Q}, \mathcal{P}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_1}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}).$$

Finally, in case of $J_0 = J$, we have $\mathcal{O}' \subset \mathcal{O}$ and $\iota f = f|_{C_c^\infty(\mathcal{O}')}$ for all $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$. \blacktriangleleft

Proof. By assumption, there is $N := k(\mathcal{P}_{J_0}, \mathcal{Q}) \in \mathbb{N}_0$ such that for every $j \in J_0$, we have $P_j \subset Q_{i_j}^{N*}$ for some $i_j \in I$. Now, we want to apply Theorem 5.6 with the following choices:

$$K := I_0 := I, \quad I^{(k)} := k^{(2N+2)*}, \quad J^{(k)} := J_0 \cap J_k \quad \text{and} \quad q_k := p_1$$

for all $k \in K = I$. After verifying the assumptions of Theorem 5.6 with these choices, we will show that the map ι from that theorem coincides with the map ι as defined in the present corollary (with the special case for $J_0 = J$).

We first verify condition (5.3). Thus, assume to the contrary that there is some $j \in J^{(k)} = J_0 \cap J_k$ and some $i \in I \setminus I^{(k)}$ with $Q_i \cap P_j \neq \emptyset$. Because of $j \in J_0$, we have $\emptyset \neq Q_i \cap P_j \subset Q_i \cap Q_{i_j}^{N*}$ and thus $i \in i_j^{(N+1)*}$. Furthermore, $j \in J_0 \cap J_k$ implies $\emptyset \neq P_j \cap Q_k \subset Q_{i_j}^{N*} \cap Q_k$ and hence $k \in i_j^{(N+1)*}$, i.e. $i_j \in k^{(N+1)*}$, which finally yields $i \in i_j^{(N+1)*} \subset k^{(2N+2)*} = I^{(k)}$, in contradiction to $i \in I \setminus I^{(k)}$. We have thus verified condition (5.3).

Next, we want to verify conditions (5.6) and (5.7) with $X := Y_{1/\theta} \leq \mathbb{C}^I$, where $\theta_i := |\det T_i|^{\left(\frac{1}{p_1}-1\right)_+}$. Note that equation (3.11) shows that $(|\det T_i|)_{i \in I}$ is \mathcal{Q} -moderate with $C_{|\det T_i|, \mathcal{Q}} \leq C_{\mathcal{Q}}^d$, so that also $1/\theta$ is \mathcal{Q} -moderate with $C_{\theta, \mathcal{Q}} \leq C_{\mathcal{Q}}^{d(p_1^{-1}-1)_+}$. By Lemma 4.13, this implies that X is \mathcal{Q} -regular, with

$$\|\Gamma_{\mathcal{Q}}\|_{X \rightarrow X} \leq C_{\mathcal{Q}}^{d(p_1^{-1}-1)_+} \cdot \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y} =: C_1.$$

Note that we have $I^{(k)} \supset \{k\}$ and hence $L = \bigcup_{k \in K} I^{(k)} = I$. Thus, condition (5.7) requires boundedness of the embedding

$$\eta_2 : Y \xrightarrow{\iota} X \left(\left[\ell_v^{q_k} (I^{(k)}) \right]_{k \in K} \right) = X \left(\left[\ell_v^{p_1} (k^{(2N+2)*}) \right]_{k \in K} \right),$$

with

$$\begin{aligned} v_{k,i} &= \begin{cases} |\det T_i|^{\frac{1}{p_1} - \frac{1}{q_k}} \cdot \left[\sup_{j \in J^{(k)}} \lambda(\overline{P_j} - \overline{Q_i}) \right]^{\frac{1}{q_k} - 1}, & \text{if } q_k < 1, \\ |\det T_i|^{\frac{1}{p_1} - \frac{1}{q_k}}, & \text{if } q_k \geq 1 \end{cases} \\ &= \begin{cases} \left[\sup_{j \in J^{(k)}} \lambda(\overline{P_j} - \overline{Q_i}) \right]^{\frac{1}{p_1} - 1}, & \text{if } p_1 < 1, \\ 1, & \text{if } p_1 \geq 1 \end{cases} \end{aligned}$$

for $k \in K = I$ and $i \in I^{(k)} = k^{(2N+2)*}$.

We now estimate this weight further in case of $p_1 < 1$. For $j \in J^{(k)} = J_0 \cap J_k$, Lemma 2.11 yields $P_j \subset Q_k^{(2N+2)*} \subset Q_i^{(4N+4)*}$, so that we get

$$\lambda(\overline{P_j} - \overline{Q_i}) \leq \lambda(\overline{Q_i^{(4N+4)*}} - \overline{Q_i^{(4N+4)*}}) \leq C_2 \cdot |\det T_i|$$

for some constant $C_2 = C_2(\mathcal{Q}, N, d) \geq 1$, cf. Corollary 2.8. All in all, we have shown

$$v_{k,i} \leq \begin{cases} C_2 \cdot |\det T_i|^{\frac{1}{p_1} - 1} = C_2 \cdot \theta_i, & \text{if } p_1 < 1, \\ 1 \leq C_2 \cdot \theta_i, & \text{if } p_1 \geq 1. \end{cases}$$

Thus, Lemma 5.10 yields (because of $Y = X_\theta$ and $K = I$) that

$$\begin{aligned} Y &\xrightarrow{\|\cdot\| \leq C_3 = C_3(N, p_1, N_{\mathcal{Q}}, \|\Gamma_{\mathcal{Q}}\|_{X \rightarrow X})} X \left(\left[\ell_\theta^{p_1} (k^{(2N+2)*}) \right]_{k \in K} \right) \\ &\xrightarrow{\|\cdot\| \leq C_2 \text{ since } v_{k,i} \leq C_2 \cdot \theta_i} X \left(\left[\ell_v^{p_1} (k^{(2N+2)*}) \right]_{k \in K} \right) \\ &= X \left(\left[\ell_v^{q_k} (I^{(k)}) \right]_{k \in K} \right). \end{aligned}$$

All in all, this shows that η_2 is bounded with $\|\eta_2\| \leq C_2 C_3$, so that condition (5.7) is satisfied.

Verification of condition (5.6) is easier: First note that each $j \in J_0$ satisfies $\emptyset \neq P_j \subset Q_{i_j}^{N^*}$ for some $i_j \in I$. In particular, $P_j \cap Q_k \neq \emptyset$ for some $k \in K = I$, so that we get $j \in J_0 \cap J_k = J^{(k)}$. Hence—in the notation of Theorem 5.6—we have $J_{00} = \bigcup_{k \in K} J^{(k)} = J_0$. Thus, (because of $X = Y_{1/\theta}$), condition (5.6) precisely requires boundedness of

$$\eta_1 : Y \left(\left[\ell_{(w_{i,j}/\theta_i)}^{p_1^\Delta} (J^{(i)}) \right]_{i \in I} \right) = X \left(\left[\ell_w^{q_k^\Delta} (J^{(k)}) \right]_{k \in K} \right) \xrightarrow{!} Z, (x_j)_{j \in J_0} \mapsto (x_j)_{j \in J} \text{ with } x_j = 0 \text{ for } j \in J \setminus J_0,$$

where (cf. equation (5.4))

$$\frac{w_{i,j}}{\theta_i} = \begin{cases} |\det S_j|^{\frac{1}{p_2}-1} \cdot |\det T_i|^{1-\frac{1}{p_1}} = u_i, & \text{if } p_1 < 1, \\ |\det S_j|^{\frac{1}{p_2}-\frac{1}{p_1}} = u_i, & \text{if } p_1 \geq 1. \end{cases}$$

Thus, a moment's thought shows that we have $\|\eta_1\| = \|\eta\|$, with η as in the statement of the present corollary.

Now, Theorem 5.6 shows that

$$\iota^{(0)} : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z), f \mapsto \sum_{(i,j) \in I \times J_0} \varphi_i \psi_j f$$

is well-defined and bounded with absolute convergence of the defining series and with

$$\|\iota^{(0)}\| \leq C_4 \cdot \|\eta_1\| \cdot \|\eta_2\| \leq C_2 C_3 C_4 \cdot \|\eta\|$$

for some constant $C_4 = C_4(d, p_1, p_2, \mathcal{Q}, \mathcal{P}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_1}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z})$.

But for $j \in J_0$, we have $P_j \subset Q_{i_j}^{N^*}$ for some $i_j \in I$. Because of $Q_{i_j}^{N^*} \subset \mathcal{O}$ (cf. Lemma 2.4), this yields $\psi_j \in C_c^\infty(\mathcal{O})$, so that $\psi_j \cdot f \in \mathcal{S}'(\mathbb{R}^d)$ is well-defined for all $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \leq \mathcal{D}'(\mathcal{O})$. Finally, since $(\varphi_i)_{i \in I}$ is a (locally finite) partition of unity on \mathcal{O} , we get $\psi_j = \sum_{i \in I} \varphi_i \psi_j$ and thus (using the absolute convergence of the series)

$$\langle \iota^{(0)} f, g \rangle = \sum_{j \in J_0} \sum_{i \in I} \langle f, \varphi_i \psi_j g \rangle = \sum_{j \in J_0} \langle f, \psi_j g \rangle = \left\langle \sum_{j \in J_0} \psi_j f, g \right\rangle$$

for all $g \in C_c^\infty(\mathcal{O}')$. Hence, $\iota^{(0)} f = \iota f$ with ι as in the statement of the present corollary. In particular, this implies absolute convergence of the series defining $\langle \iota f, g \rangle$ for all $g \in C_c^\infty(\mathcal{O})$.

Finally, assume $J = J_0$. We just saw $P_j \subset \mathcal{O}$ for all $j \in J_0$, so that $\mathcal{O}' = \bigcup_{j \in J} P_j \subset \mathcal{O}$. Now, for $g \in C_c^\infty(\mathcal{O}') \subset C_c^\infty(\mathcal{O})$, we have $g = \sum_{j \in J} \psi_j g$ with only finitely many terms not vanishing, since $(\psi_j)_{j \in J}$ is a locally finite partition of unity on \mathcal{O}' , cf. Lemma 2.4. Hence,

$$\langle \iota f, g \rangle = \sum_{j \in J} \langle f, \psi_j g \rangle = \langle f, g \rangle = \langle f|_{C_c^\infty(\mathcal{O}')} , g \rangle,$$

as desired. \square

As for Corollary 5.8 above, one can greatly simplify the process of verifying boundedness of the embedding η from above, at least if Y, Z are both weighted Lebesgue sequence spaces:

Corollary 5.12. *Let $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ be two admissible coverings of the open sets $\emptyset \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$, respectively. Furthermore, let $w = (w_i)_{i \in I}$ and $u^{(1)} = (u_i^{(1)})_{i \in I}$ be \mathcal{Q} -moderate and let $v = (v_j)_{j \in J}$ and $u^{(2)} = (u_j^{(2)})_{j \in J}$ be \mathcal{P} -moderate. Finally, let $q_1, q_2, r \in (0, \infty]$ and define $u := (u_i^{(1)} u_j^{(2)})_{i \in I, j \in J}$.*

If $J_0 \subset J$ is chosen so that $\mathcal{P}_{J_0} = (P_j)_{j \in J_0}$ is almost subordinate to \mathcal{Q} , then we have

$$\|\eta\| \asymp \left\| \left(w_i^{-1} \cdot \left\| (v_j / u_{i,j})_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2 \cdot (r/q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \quad (5.23)$$

for

$$\eta : \ell_w^{q_1} \left([\ell_u^r (J_0 \cap J_i)]_{i \in I} \right) \hookrightarrow \ell_v^{q_2} (J_0).$$

Precisely, this means that η is well-defined and bounded if and only if the right-hand side of equation (5.23) is finite. Furthermore, there is a constant $C \geq 1$ depending only on r, q_1, q_2 and on $C_{w, \mathcal{Q}}, N_{\mathcal{Q}}, k(\mathcal{P}_{J_0}, \mathcal{Q}), C_{u^{(1)}, \mathcal{Q}}$ which satisfies

$$C^{-1} \cdot M \leq \|\eta\| \leq C \cdot M \quad \text{for} \quad M := \left\| \left(w_i^{-1} \cdot \left\| (v_j / u_{i,j})_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2 \cdot (r/q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}.$$

Finally, if

- $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ and $\mathcal{P} = (S_j P'_j + c_j)_{j \in J}$ are tight semi-structured coverings,
- \mathcal{P}_{J_0} is relatively \mathcal{Q} -moderate,
- there is some $s \in \mathbb{N}_0$ and some $C_0 > 0$ such that

$$\lambda(Q_i) \leq C_0 \cdot \lambda \left(\bigcup_{j \in J_0 \cap J_i} P_j^{s*} \right)$$

holds for all $i \in I$ with $1 \leq |J_0 \cap J_i| < \infty$,

- $u^{(2)}|_{J_0}$ and $v|_{J_0}$ are relatively \mathcal{Q} -moderate and
- for each $i \in I^{(0)} := \{i \in I \mid J_0 \cap J_i \neq \emptyset\}$, some $j_i \in J_0 \cap J_i$ is selected,

then

$$\left\| \left(w_i^{-1} \cdot \left\| (v_j / u_{i,j})_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2 \cdot (r/q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \asymp \left\| \left(\frac{v_{j_i}}{w_i \cdot u_{i,j_i}} \cdot [\det T_i / |\det S_{j_i}|] \left(\frac{1}{q_2} - \frac{1}{r} \right)_+ \right)_{i \in I^{(0)}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} ,$$

where the implied constant only depends on

$$d, s, r, q_2, C_0, k(\mathcal{P}_{J_0}, \mathcal{Q}), C_{\text{mod}}(\mathcal{P}_{J_0}, \mathcal{Q}), \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{Q}}, \varepsilon_{\mathcal{P}}, C_{u^{(2)}|_{J_0}, \mathcal{P}, \mathcal{Q}}, C_{v|_{J_0}, \mathcal{P}, \mathcal{Q}} \quad \blacktriangleleft$$

Remark 5.13. In particular, if we have $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$ in Corollary 5.11, then the embedding η from that corollary satisfies

$$\|\eta\| \asymp \left\| \left(w_i^{-1} \cdot \left\| (v_j / u_{i,j})_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2 \cdot (p_1^\Delta / q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1 / q_2)'}} =: M$$

with $u_{i,j} = \begin{cases} |\det S_j|^{\frac{1}{p_2} - 1} \cdot |\det T_i|^{1 - \frac{1}{p_1}}, & \text{if } p_1 < 1, \\ |\det S_j|^{\frac{1}{p_2} - \frac{1}{p_1}}, & \text{if } p_1 \geq 1, \end{cases}$

where the implied constant only depends on $p_1, q_1, q_2, C_{w, \mathcal{Q}}, \mathcal{Q}, k(\mathcal{P}_{J_0}, \mathcal{Q})$. Here, we used the preceding corollary with

$$u_i^{(1)} := \begin{cases} |\det T_i|^{1 - \frac{1}{p_1}}, & \text{if } p_1 < 1, \\ 1, & \text{if } p_1 \geq 1 \end{cases} \quad \text{and} \quad u_j^{(2)} := \begin{cases} |\det S_j|^{\frac{1}{p_2} - 1}, & \text{if } p_1 < 1, \\ |\det S_j|^{\frac{1}{p_2} - \frac{1}{p_1}}, & \text{if } p_1 \geq 1. \end{cases}$$

Finally, if the additional assumptions from the second part of the corollary are satisfied (where relative \mathcal{Q} -moderateness of $u^{(2)}|_{J_0}$ is implied by relative \mathcal{Q} -moderateness of \mathcal{P}_{J_0}), we get

$$M \asymp \left\| \left(\frac{v_{j_i}}{w_i} \cdot |\det S_{j_i}|^{\frac{1}{p_1} - \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\Delta}} \right)_+} \cdot |\det T_i|^{\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\Delta}} \right)_+} \right)_{i \in I^{(0)}} \right\|_{\ell^{q_2 \cdot (q_1 / q_2)'}} \quad (5.24)$$

with $1/p_1^{\pm\Delta} = \min \left\{ \frac{1}{p_1}, 1 - \frac{1}{p_1} \right\}$, i.e. with

$$p_1^{\pm\Delta} := \begin{cases} p_1, & \text{if } p_1 \geq 2, \\ \frac{p_1}{p_1 - 1}, & \text{if } 0 < p_1 < 2 \text{ and } p_1 \neq 1, \\ \infty, & \text{if } p_1 = 1. \end{cases}$$

To see this, note that $p_1^\Delta = p_1^{\pm\Delta}$ for $p_1 \in [1, \infty]$, while $p_1^{\pm\Delta}$ is negative for $0 < p_1 < 1$. Furthermore, note that if we define the **duality defect** p_d of $p \in (0, \infty]$ by

$$p_d := \min \left\{ 0, 1 - \frac{1}{p} \right\} = \begin{cases} 1 - \frac{1}{p}, & \text{if } p < 1, \\ 0, & \text{if } p \geq 1, \end{cases}$$

then $1/p_1^{\pm\Delta} = 1/p_1^\Delta + p_d$, as a simple case distinction shows. Additionally, we have $u_i^{(1)} = |\det T_i|^{(p_1)_d}$ and $u_j^{(2)} = |\det S_j|^{\frac{1}{p_2} - \frac{1}{p_1} - (p_1)_d}$ for all $i \in I$ and $j \in J$, so that we get

$$u_{i,j_i}^{-1} \cdot [|\det T_i| / |\det S_{j_i}|]^{\left(\frac{1}{q_2} - \frac{1}{p_1^\Delta}\right)_+} = |\det T_i|^{\left(\frac{1}{q_2} - \frac{1}{p_1^\Delta}\right)_+ - (p_1)_d} \cdot |\det S_{j_i}|^{\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{q_2} - \frac{1}{p_1^\Delta}\right)_+ + (p_1)_d}.$$

Finally, note

$$\left(\frac{1}{q_2} - \frac{1}{p_1^\Delta}\right)_+ - (p_1)_d = \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\Delta}}\right)_+.$$

In case of $p_1 \in [1, \infty]$, this is clear. But for $p_1 \in (0, 1)$, we have $p_1^\Delta = \infty$ and hence

$$\left(\frac{1}{q_2} - \frac{1}{p_1^\Delta}\right)_+ - (p_1)_d = \frac{1}{q_2} - (p_1)_d = \frac{1}{q_2} - 1 + \frac{1}{p_1} = \frac{1}{q_2} - \frac{1}{p_1^{\pm\Delta}}.$$

Conversely, since $p_1^{\pm\Delta} < 0$ for $p_1 \in (0, 1)$, we have

$$\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\Delta}}\right)_+ = \frac{1}{q_2} - \frac{1}{p_1^{\pm\Delta}}.$$

All in all, these considerations establish equation (5.24), where the implied constant only depends on

$$d, s, p_1, p_2, q_2, C_0, k(\mathcal{P}_{J_0}, \mathcal{Q}), C_{\text{mod}}(\mathcal{P}_{J_0}, \mathcal{Q}), \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{Q}}, \varepsilon_{\mathcal{P}}, C_{v|_{J_0}, \mathcal{P}, \mathcal{Q}}. \quad \blacklozenge$$

Proof of Corollary 5.12. For the proof of the first part, we want to use the estimate provided by Corollary 4.12 with $K = I$, $I = J_0$, $X = \ell_w^{q_1}(K) = \ell_w^{q_1}(I)$ and

$$I^{(k, \mathfrak{h})} := I^{(k)} := J_0 \cap J_k = \{j \in J_0 \mid P_j \cap Q_k \neq \emptyset\}$$

for $k \in K = I$ to calculate the norm of this embedding. For this, we first have to verify the assumptions of Corollary 4.12, which are just the assumptions of Lemma 4.9.

To this end, first note that we have $J_0 = \bigcup_{k \in K} I^{(k)}$. Indeed, “ \supset ” is trivial; for the reverse inclusion, let $j \in J_0$ be arbitrary. Since \mathcal{P}_{J_0} is almost subordinate to \mathcal{Q} , we have $P_j \subset \mathcal{O}$ and hence $P_j \cap Q_i \neq \emptyset$ for some $i \in I = K$. Hence, $j \in J_0 \cap J_i = I^{(i)}$.

Now, define a relation \sim on $K = I$ by $i \sim \ell \iff I^{(i)} \cap I^{(\ell)} \neq \emptyset$. For brevity, let us write $n := k(\mathcal{P}_{J_0}, \mathcal{Q}) \in \mathbb{N}_0$, which is well-defined, since \mathcal{P}_{J_0} is almost subordinate to \mathcal{Q} . Lemma 4.11 (with $n = 0$ and $\mathcal{P}_{J_0}, \mathcal{Q}$ for $(Q_i)_{i \in I}, \mathcal{R}$) implies that the classes $[i]$ of this relation satisfy

$$[i] \subset i^{(4n+5)*}, \quad \text{as well as} \quad |[i]| \leq N_{\mathcal{Q}}^{4n+5} =: N \quad \forall i \in I.$$

In particular, condition (4.5) of Lemma 4.9 is satisfied. Furthermore, the generalized clustering map Θ from that lemma satisfies

$$|(\Theta x)_i| \leq \sum_{\ell \in [i]} |x_\ell| \leq \sum_{\ell \in i^{(4n+5)*}} |x_\ell| = (\Theta_{4n+5} |x|)_i \quad \forall x = (x_i)_{i \in I} \in X = \ell_w^{q_1}(I)$$

for all $i \in I$, with Θ_n (for $n \in \mathbb{N}_0$) as in Lemma 3.9. Thus, by solidity of $X = \ell_w^{q_1}$, we see that Θ is well-defined and bounded with

$$\|\Theta\| \leq \|\Theta_{4n+5}\| \leq \|\Gamma_{\mathcal{Q}}\|_{\ell_w^{q_1} \rightarrow \ell_w^{q_1}}^{4n+5} \leq \left(C_{w, \mathcal{Q}} \cdot N_{\mathcal{Q}}^{1+\frac{1}{q_1}}\right)^{4n+5},$$

cf. Lemma 4.13.

For the final assumption of Lemma 4.9 (regarding the weight $u = (u_{i,j})$) let $i, \ell \in K = I$ and $j \in I^{(i)} \cap I^{(\ell)}$. In particular, this implies $i \in [i] \subset \ell^{(4n+5)*}$, so that Lemma 2.9 yields

$$\begin{aligned} u_{i,j} &= u_i^{(1)} \cdot u_j^{(2)} \\ &\leq C_{u^{(1)}, \mathcal{Q}}^{4n+5} \cdot u_\ell^{(1)} \cdot u_j^{(2)} \\ &= C_{u^{(1)}, \mathcal{Q}}^{4n+5} \cdot u_{\ell,j}. \end{aligned}$$

Hence, we can choose (in the notation of Lemma 4.9) $C_u = C_{u^{(1)}, \mathcal{Q}}^{4n+5}$.

Now, we can finally apply parts (1) and (3) of Corollary 4.12, which yield

$$\|\eta\| \asymp \left\| \left(w_i^{-1} \cdot \left\| (v_j/u_{i,j})_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2 \cdot (r/q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}$$

where the implied constant only depends on

$$\|\Theta\| \leq \left(C_{w, \mathcal{Q}} \cdot N_{\mathcal{Q}}^{1+q_1^{-1}} \right)^{4n+5}, \quad N = N_{\mathcal{Q}}^{4n+5}, \quad C_u = C_{u^{(1)}, \mathcal{Q}}^{4n+5} \text{ and } r, q_2.$$

For the final part of the corollary, note that the present assumptions include the prerequisites of Lemma 2.17 (with interchanged roles of \mathcal{Q}, \mathcal{P} and with $I_0 = J_0$), so that we get

$$|J_0 \cap J_i| \asymp |\det T_i| / |\det S_{j_i}|$$

for all $i \in I$ with $J_0 \cap J_i \neq \emptyset$, i.e. for all $i \in I^{(0)}$. Here, the implied constant only depends on those quantities which are mentioned in the second part of the present corollary.

Furthermore, relative \mathcal{Q} -moderateness of $u^{(2)}|_{J_0}$ and of $v|_{J_0}$ implies

$$\frac{v_j}{u_{i,j}} = \frac{v_j}{u_i^{(1)} u_j^{(2)}} \leq C_{u^{(2)}|_{J_0}, \mathcal{P}_{J_0}, \mathcal{Q}} C_{v|_{J_0}, \mathcal{P}_{J_0}, \mathcal{Q}} \cdot \frac{v_{j_i}}{u_i^{(1)} u_{j_i}^{(2)}} = C_{u^{(2)}|_{J_0}, \mathcal{P}_{J_0}, \mathcal{Q}} C_{v|_{J_0}, \mathcal{P}_{J_0}, \mathcal{Q}} \cdot \frac{v_{j_i}}{u_{i,j_i}}$$

and likewise

$$\frac{v_{j_i}}{u_{i,j_i}} \leq C_{u^{(2)}|_{J_0}, \mathcal{P}_{J_0}, \mathcal{Q}} C_{v|_{J_0}, \mathcal{P}_{J_0}, \mathcal{Q}} \cdot \frac{v_j}{u_{i,j}}$$

for all $i \in I^{(0)}$ and $j \in J_0 \cap J_i$. All in all, we derive

$$\left\| (v_j/u_{i,j})_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2 \cdot (r/q_2)'}} \asymp |J_0 \cap J_i|^{[q_2 \cdot (r/q_2)']^{-1}} \cdot \frac{v_{j_i}}{u_{i,j_i}} \asymp [|\det T_i| / |\det S_{j_i}|]^{(\frac{1}{q_2} - \frac{1}{r})_+} \cdot \frac{v_{j_i}}{u_{i,j_i}}$$

for all $i \in I^{(0)}$, with the implied constants as in the statement of the corollary. Here, we used the formula $\frac{1}{q \cdot (r/q)'} = \left(\frac{1}{q} - \frac{1}{r} \right)_+$ (cf. equation (4.2)) in the last step.

Since all indices $i \in I \setminus I^{(0)}$ can be neglected in the left-hand side of the desired estimate, this completes the proof. \square

As a further corollary of Theorem 5.6, we derive a result which applies to coverings \mathcal{Q}, \mathcal{P} which exhibit a kind of “mixed” subordinateness. Roughly, we assume that we can write $\mathcal{O} \cap \mathcal{O}' = A \cup B$, such that \mathcal{Q} is almost subordinate to \mathcal{P} “near A ” and vice versa “near B ”. We remark that the following corollary is a generalized version of [23, Corollary 5.1.11] from my PhD thesis.

Corollary 5.14. *Let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J} = (S_j P'_j + c_j)_{j \in J}$ be two semi-structured coverings of the open sets $\mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$. Let $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ be \mathcal{Q} -regular and \mathcal{P} -regular sequence spaces, respectively and let $p_1, p_2 \in (0, \infty]$ with $p_1 \leq p_2$. Assume that there are L^{p_1} -BAPUs $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_j)_{j \in J}$ for \mathcal{Q} and \mathcal{P} , respectively.*

Assume that there are subsets $A, B \subset \mathbb{R}^d$ with the following properties:

- (1) *We have $\mathcal{O} \cap \mathcal{O}' = A \cup B$.*
- (2) *With*

$$I_A := \{i \in I \mid Q_i \cap A \neq \emptyset\},$$

the family $\mathcal{Q}_{I_A} = (Q_i)_{i \in I_A}$ is almost subordinate to \mathcal{P} .

- (3) *With*

$$J_B := \{j \in J \mid P_j \cap B \neq \emptyset\},$$

the family $\mathcal{P}_{J_B} = (P_j)_{j \in J_B}$ is almost subordinate to \mathcal{Q} .

Then, we have $L := \bigcup_{j \in J \setminus J_B} I_j \subset I_A$.

Finally, define

$$\theta_i := |\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}} \quad \text{and} \quad u_{i,j} := \begin{cases} |\det S_j|^{\frac{1}{p_2} - 1} \cdot |\det T_i|^{1 - \frac{1}{p_1}}, & \text{if } p_1 < 1, \\ |\det S_j|^{\frac{1}{p_2} - \frac{1}{p_1}}, & \text{if } p_1 \geq 1, \end{cases}$$

for $i \in I$ and $j \in J$ and assume that the maps

$$\beta_1 : Y \hookrightarrow Z|_{J \setminus J_B} \left(\left[\ell_\theta^{p_2^\vee}(I_j) \right]_{j \in J \setminus J_B} \right), (x_i)_{i \in I} \mapsto (x_i)_{i \in L} \text{ with } L = \bigcup_{j \in J \setminus J_B} I_j,$$

$$\beta_2 : Y \left(\left[\ell_u^{p_1^\Delta}(J_i \cap J_B) \right]_{i \in I} \right) \hookrightarrow Z|_{J_B},$$

are bounded. Here, as usual $I_j = \{i \in I \mid Q_i \cap P_j \neq \emptyset\}$ and $J_i = \{j \in J \mid P_j \cap Q_i \neq \emptyset\}$.

Then, the map

$$\iota : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z), f \mapsto \sum_{i \in I} \varphi_i f$$

is well-defined and bounded with $\|\iota\| \leq C \cdot (\|\beta_1\| + \|\beta_2\|)$ for some constant

$$C = C(d, p_1, p_2, C_Z, \mathcal{Q}, \mathcal{P}, k(\mathcal{P}_{J_B}, \mathcal{Q}), k(\mathcal{Q}_{I_A}, \mathcal{P}), C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_1}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z})$$

and with the following additional properties:

(1) For $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$ and $g \in C_c^\infty(\mathcal{O} \cap \mathcal{O}')$, we have

$$\langle \iota f, g \rangle = \langle f, g \rangle.$$

In particular, if $\mathcal{O} = \mathcal{O}'$, then $\iota f = f$ for all $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \leq \mathcal{D}'(\mathcal{O})$.

(2) If $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$ is given by (integration against) a measurable function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ with $f \in L_{\text{loc}}^1(\mathcal{O} \cup \mathcal{O}')$ and with $f = 0$ almost everywhere on $\mathcal{O}' \setminus \mathcal{O}$, then $\iota f = f$.

(3) In particular, $\iota f = f$ for $f \in C_c^\infty(\mathcal{O} \cap \mathcal{O}') \cap \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$.

(4) For $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$, we have $\text{supp } \iota f \subset \mathcal{O}' \cap \text{supp } f$.

Finally, if $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$ for certain $q_1, q_2 \in (0, \infty]$ and weights $w = (w_i)_{i \in I}$ and $v = (v_j)_{j \in J}$ which are \mathcal{Q} -moderate and \mathcal{P} -moderate, respectively, then

$$\|\beta_1\| \asymp \left\| \left(v_j \cdot \left\| \left(w_i^{-1} \cdot |\det T_i|^{p_1^{-1} - p_2^{-1}} \right)_{i \in I_j} \right\|_{\ell^{p_2^\vee \cdot (q_1/p_2^\vee)'}} \right)_{j \in J \setminus J_B} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \quad (5.25)$$

$$\|\beta_2\| \asymp \left\| \left(w_i^{-1} \cdot \left\| (v_j / u_{i,j})_{j \in J_i \cap J_B} \right\|_{\ell^{q_2 \cdot (p_1^\Delta/q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \quad (5.26)$$

where the implied constants only depend on

$$d, p_1, p_2, q_1, q_2, \mathcal{Q}, \mathcal{P}, k(\mathcal{Q}_{I_A}, \mathcal{P}), k(\mathcal{P}_{J_B}, \mathcal{Q}), C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}. \quad \blacktriangleleft$$

Remark. Note that the corollary in particular yields $I_j \subset L \subset I_A$ for all $j \in J \setminus J_B$. Thus, under the assumptions of the last part of the corollary, we have

$$\begin{aligned} \|\beta_1\| &\asymp \left\| \left(v_j \cdot \left\| \left(w_i^{-1} \cdot |\det T_i|^{p_1^{-1} - p_2^{-1}} \right)_{i \in I_j} \right\|_{\ell^{p_2^\vee \cdot (q_1/p_2^\vee)'}} \right)_{j \in J \setminus J_B} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ &\leq \left\| \left(v_j \cdot \left\| \left(w_i^{-1} \cdot |\det T_i|^{p_1^{-1} - p_2^{-1}} \right)_{i \in I_j \cap I_A} \right\|_{\ell^{p_2^\vee \cdot (q_1/p_2^\vee)'}} \right)_{j \in J} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}, \end{aligned}$$

which is more similar to the simplified form given for $\|\beta_2\|$. Note, though, that we only get an *upper* bound for $\|\beta_1\|$. \blacklozenge

Proof. To begin with, we establish $L \subset I_A$. Indeed, if $i \in I_j$ for some $j \in J \setminus J_B$, there is some $\xi \in Q_i \cap P_j \subset \mathcal{O} \cap \mathcal{O}' = A \cup B$. In case of $\xi \in B$, we would have $\xi \in P_j \cap B$ and hence $j \in J_B$, a contradiction. Hence, $\xi \in A$, i.e. $\xi \in Q_i \cap A \neq \emptyset$ and hence $i \in I_A$.

Now, let us begin with the actual proof. Below, we will use Theorem 5.6 to show that the following two maps are well-defined and bounded:

$$\begin{aligned} \iota_1 : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) &\rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z), f \mapsto \sum_{(i,j) \in I_A \times (J \setminus J_B)} \varphi_i \psi_j f, \\ \iota_2 : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) &\rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z), f \mapsto \sum_{(i,j) \in I \times J_B} \varphi_i \psi_j f, \end{aligned}$$

both with absolute convergence of the series defining $\langle \iota_1 f, g \rangle$ and $\langle \iota_2 f, g \rangle$ for $g \in C_c^\infty(\mathcal{O}')$. Let us assume for the moment that this holds.

We first show that $\iota_1 f = \sum_{(i,j) \in I \times (J \setminus J_B)} \varphi_i \psi_j f$ for $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$. To this end, it suffices to show $\varphi_i \psi_j \equiv 0$ for $i \in I \setminus I_A$ and $j \in J \setminus J_B$. But $\varphi_i \psi_j \not\equiv 0$ would imply $i \in I_j \subset L \subset I_A$, since $j \in J \setminus J_B$. Since this is impossible for $i \in I \setminus I_A$, we have shown $\varphi_i \psi_j \equiv 0$ for $i \in I \setminus I_A$ and $j \in J \setminus J_B$, as needed.

Altogether, we see

$$\begin{aligned} \iota_1 f + \iota_2 f &= \sum_{(i,j) \in I \times (J \setminus J_B)} \varphi_i \psi_j f + \sum_{(i,j) \in I \times J_B} \varphi_i \psi_j f \\ &= \sum_{(i,j) \in I \times J} \varphi_i \psi_j f, \end{aligned}$$

again with absolute convergence of the defining series, since all we did was to add some vanishing terms and to add two absolutely convergent (defining) series.

But for $g \in C_c^\infty(\mathcal{O}')$, we have $g = \sum_{j \in J} \psi_j g$, with only finitely many non-vanishing terms, since $(\psi_j)_{j \in J}$ is a locally finite partition of unity on \mathcal{O}' , cf. Lemma 2.4. This implies

$$\langle \iota_1 f + \iota_2 f, g \rangle = \sum_{i \in I} \sum_{j \in J} \langle \varphi_i f, \psi_j g \rangle = \sum_{i \in I} \langle \varphi_i f, g \rangle,$$

with absolute convergence of the series. Hence, we see that ι as defined in the present corollary is well-defined and bounded, with absolute convergence of the defining series.

It remains to establish the additional properties of ι (and boundedness of ι_1, ι_2). Validity of property (1) is shown just as above: For $g \in C_c^\infty(\mathcal{O} \cap \mathcal{O}')$, we have $g = \sum_{i \in I} \varphi_i g$ (with only finitely many terms not vanishing) and this easily yields $\langle \iota f, g \rangle = \langle f, g \rangle$.

Property (3) is an immediate consequence of property (2), which we prove next: For each $\xi \in \mathcal{O}$, there is some $i_\xi \in I$ with $\varphi_{i_\xi}(\xi) \neq 0$ and hence $\xi \in Q_{i_\xi}^\circ$. Since \mathcal{O} is second countable, there is thus a sequence $(i_{\xi_n})_{n \in \mathbb{N}}$ satisfying $\mathcal{O} = \bigcup_{n \in \mathbb{N}} Q_{i_{\xi_n}}^\circ$. But this implies $\varphi_i \equiv 0$ for all $i \in I \setminus I_0$, with the countable set $I_0 := \bigcup_{n \in \mathbb{N}} i_{\xi_n}^*$. Indeed, if $\varphi_i \not\equiv 0$, then $\varphi_i(\xi) \neq 0$ for some $\xi \in Q_i \subset \mathcal{O}$. By what we just saw, $\xi \in Q_{i_{\xi_n}}$ for some $n \in \mathbb{N}$, so that we get $Q_i \cap Q_{i_{\xi_n}} \neq \emptyset$ and hence $i \in i_{\xi_n}^* \subset I_0$.

Now, recall that Fourier inversion implies

$$|\varphi_i(\xi)| = \left| \widehat{\mathcal{F}^{-1} \varphi_i}(\xi) \right| \leq \|\mathcal{F}^{-1} \varphi_i\|_{L^1} \leq C_\Phi < \infty$$

for all $\xi \in \mathbb{R}^d$ and $i \in I$. As we saw in the previous paragraph, for each $\xi \in \mathcal{O}$, there is some $n = n_\xi \in \mathbb{N}$ with $\varphi_i(\xi) = 0$ for $i \in I \setminus i_{\xi_n}^*$. This implies

$$\sum_{i \in I_0} |\varphi_i(\xi)| = \sum_{i \in i_{\xi_n}^*} |\varphi_i(\xi)| \leq C_\Phi |i_{\xi_n}^*| \leq C_\Phi N_{\mathcal{Q}} < \infty$$

for all $\xi \in \mathcal{O}$. For $\xi \in \mathbb{R}^d \setminus \mathcal{O}$, the left-hand side vanishes, so that the estimate is true for all $\xi \in \mathbb{R}^d$.

Now, let $g \in C_c^\infty(\mathcal{O}')$ be arbitrary and set $K := \text{supp } g$. Note that $K \subset \mathcal{O} \cup \mathcal{O}'$ is compact, so that $\mathbb{1}_K \cdot f \in L^1(\mathbb{R}^d)$, since $f \in L^1_{\text{loc}}(\mathcal{O} \cup \mathcal{O}')$. Furthermore,

$$\sum_{i \in I_0} |\varphi_i(\xi) f(\xi) g(\xi)| \leq C_\Phi N_{\mathcal{Q}} \|g\|_{\text{sup}} \cdot (\mathbb{1}_K \cdot |f|)(\xi) \in L^1(\mathbb{R}^d).$$

Thus, the dominated convergence theorem implies

$$\begin{aligned}
 \langle \iota f, g \rangle &= \sum_{i \in I} \langle \varphi_i f, g \rangle \\
 (\varphi_i &\equiv 0 \text{ for } i \in I \setminus I_0) = \sum_{i \in I_0} \int_{\mathbb{R}^d} \varphi_i(\xi) f(\xi) g(\xi) \, d\xi \\
 (I_0 \text{ countable, dominated conv.}) &= \int_{\mathbb{R}^d} \left[\sum_{i \in I_0} \varphi_i(\xi) \right] \cdot f(\xi) g(\xi) \, d\xi \\
 (\varphi_i &\equiv 0 \text{ for } i \in I \setminus I_0) = \int_{\mathbb{R}^d} \left[\sum_{i \in I} \varphi_i(\xi) \right] \cdot f(\xi) g(\xi) \, d\xi \\
 \left(\sum_{i \in I} \varphi_i &\equiv 1 \text{ on } \mathcal{O}, \varphi_i \equiv 0 \text{ on } \mathbb{R}^d \setminus \mathcal{O} \right) = \int_{\mathcal{O}} f(\xi) g(\xi) \, d\xi \\
 (f = 0 \text{ a.e. on } \mathcal{O}' \setminus \mathcal{O} \text{ and } g &= 0 \text{ on } \mathbb{R}^d \setminus \mathcal{O}') = \int_{\mathbb{R}^d} f(\xi) g(\xi) \, d\xi \\
 &= \langle f, g \rangle,
 \end{aligned}$$

as desired.

Lastly, we establish property (4): Let $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$ be arbitrary and set $A := \overline{\text{supp } f}$. Since $A \subset \mathbb{R}^d$ is closed, $U := \mathcal{O}' \setminus A$ is open. Now, let $g \in C_c^\infty(\mathcal{O}')$ with $\text{supp } g \subset U$. Because of $\text{supp } \varphi_i \subset \mathcal{O}$ for all $i \in I$, we have

$$\begin{aligned}
 \text{supp } (\varphi_i g) &\subset \mathcal{O} \cap U \\
 &= \mathcal{O} \cap (\mathcal{O}' \setminus A) \\
 &\subset \mathcal{O} \setminus A \\
 &= \mathcal{O} \setminus \overline{\text{supp } f} \\
 &\subset \mathcal{O} \setminus \text{supp } f,
 \end{aligned}$$

where we note that $\mathcal{O} \setminus \text{supp } f$ is open, since $\text{supp } f$ is closed in \mathcal{O} . Since f vanishes on the open set $\mathcal{O} \setminus \text{supp } f$, we get $\langle f, \varphi_i g \rangle = 0$ for all $i \in I$ and thus

$$\langle \iota f, g \rangle = \sum_{i \in I} \langle \varphi_i f, g \rangle = 0.$$

All in all, ιf vanishes on the open(!) set U , so that we get

$$\text{supp } (\iota f) \subset \mathcal{O}' \setminus U = \mathcal{O}' \setminus [\mathcal{O}' \setminus A] \subset \mathcal{O}' \cap A = \mathcal{O}' \cap \overline{\text{supp } f},$$

as claimed.

To complete the proof, we finally show boundedness of ι_1 and ι_2 .

Boundedness of ι_1 : Here, we apply Theorem 5.6 with $I_0 := I_A \subset I$ and $K := J_0 := J \setminus J_B$ and with

$$J^{(k)} := \{k\}, \quad \text{as well as} \quad I^{(k)} := I_k = \{i \in I \mid Q_i \cap P_k \neq \emptyset\} \quad \text{for } k \in K.$$

Recall from the beginning of the proof that we have $I^{(k)} = I_k \subset I_A = I_0$ for all $k \in K = J \setminus J_B$, as required in Theorem 5.6. Furthermore, note that we have (in the notation of Theorem 5.6)

$$J_{00} = \bigcup_{k \in K} J^{(k)} = \bigcup_{k \in K} \{k\} = K = J_0$$

and hence $J_0 \subset J_{00}$, as required.

Still in the notation of Theorem 5.6, we select $q_k := p_2 \in [p_1, p_2]$ for all $k \in K$. This entails $q^{(0)} = \inf_{k \in K} q_k = p_2$, so that $\Psi = (\psi_j)_{j \in J}$ is indeed an $L^{q^{(0)}}$ -BAPU for \mathcal{P} , as required in Theorem 5.6. Here, we used Corollary 5.4 to conclude that the L^{p_1} -BAPU Ψ is also an L^{p_2} -BAPU, since $p_1 \leq p_2$.

For the application of Theorem 5.6, we first verify condition (5.3). Thus, assume towards a contradiction that for some $k \in K = J \setminus J_B$, there is some

$$\xi \in \left(\bigcup_{i \in I_0 \setminus I^{(k)}} Q_i \right) \cap \left(\bigcup_{j \in J^{(k)}} P_j \right) = \bigcup_{i \in I_A \setminus I_k} (Q_i \cap P_k).$$

This yields some $i \in I_A \setminus I_k$ with $\xi \in Q_i \cap P_k \neq \emptyset$, in contradiction to $i \notin I_k$.

Next, we estimate the weights from equations (5.4) and (5.5). For the weight w , we have—because of $q_k = p_2$ —for $k \in K = J \setminus J_B$ and $j \in J^{(k)} = \{k\}$ that

$$w_{k,j} = w_k^{(0)} := \begin{cases} |\det S_j|^{\frac{1}{p_2}-1} = |\det S_k|^{\frac{1}{p_2}-1}, & \text{if } p_2 < 1, \\ 1, & \text{if } p_2 \geq 1. \end{cases} \quad (5.27)$$

Estimating v is more involved. Here, we first estimate for $k \in K = J \setminus J_B$ and $i \in I^{(k)} = I_k$ the quantity

$$\sup_{j \in J^{(k)}} \lambda(\overline{P_j} - \overline{Q_i}) = \lambda(\overline{P_k} - \overline{Q_i}).$$

As note above, we have $i \in I_k \subset I_A$. By assumption, this means $Q_i \subset P_{j_i}^{n*}$ for some $j_i \in J$, where $n := k(Q_{I_A}, \mathcal{P})$. But because of $i \in I_k$, we also have $\emptyset \neq P_k \cap Q_i \subset P_k \cap P_{j_i}^{n*}$ and hence $j_i \in k^{(n+1)*}$. In view of Corollary 2.8, this yields

$$\lambda(\overline{P_k} - \overline{Q_i}) \leq \lambda(\overline{P_k} - \overline{P_{j_i}^{n*}}) \leq \lambda(\overline{P_k^{(n+1)*}} - \overline{P_{j_i}^{(n+1)*}}) \leq C_1 \cdot |\det S_k|,$$

for some constant $C_1 = C_1(d, n, \mathcal{P})$.

All in all, this shows $v_{k,i} \leq C_2 \cdot v_{k,i}^{(0)}$ for all $k \in K$ and $i \in I^{(k)} = I_k$ for a suitable constant $C_2 = C_2(p_2, d, n, \mathcal{P})$ and

$$v_{k,i}^{(0)} := \begin{cases} |\det T_i|^{\frac{1}{p_1}-\frac{1}{p_2}} \cdot |\det S_k|^{\frac{1}{p_2}-1}, & \text{if } p_2 < 1, \\ |\det T_i|^{\frac{1}{p_1}-\frac{1}{p_2}}, & \text{if } p_2 \geq 1. \end{cases}$$

It remains to establish boundedness of the embeddings η_1, η_2 from equations (5.6) and (5.7), for a suitable solid sequence space $X \leq \mathbb{C}^K$. Here, we choose $X := (Z|_K)_{1/w^{(0)}}$, where we recall $K = J \setminus J_B$ and the definition of $w^{(0)} = (w_j^{(0)})_{j \in J}$ from equation (5.27).

It is not hard to see that the embedding from equation (5.6) is just $X([\ell_w^{q_k^\Delta}(J^{(k)})]_{k \in K}) \hookrightarrow Z|_{J_{00}}$, where in our case $J_{00} = J_0 = K = J \setminus J_B$. But since we have $J^{(k)} = \{k\}$ for $k \in K$ and because of $w_{k,j} = w_k^{(0)}$, we easily see

$$X([\ell_w^{q_k^\Delta}(J^{(k)})]_{k \in K}) = (Z|_K)_{w^{(0)}/w^{(0)}}([\ell_w^{q_k^\Delta}(\{k\})]_{k \in K}) = Z|_K,$$

so that the embedding η_1 from equation (5.6) is trivially bounded, with $\|\eta_1\| \leq 1$.

For the embedding η_2 from equation (5.7), first note that we have

$$\begin{aligned} v_{k,i}^{(0)}/w_k^{(0)} &= \begin{cases} |\det T_i|^{\frac{1}{p_1}-\frac{1}{p_2}} \cdot |\det S_k|^{\frac{1}{p_2}-1} / |\det S_k|^{\frac{1}{p_2}-1}, & \text{if } p_2 < 1, \\ |\det T_i|^{\frac{1}{p_1}-\frac{1}{p_2}} / 1, & \text{if } p_2 \geq 1 \end{cases} \\ &= |\det T_i|^{\frac{1}{p_1}-\frac{1}{p_2}} = \theta_i \end{aligned}$$

for all $k \in K = J \setminus J_B$ and all $i \in I^{(k)} = I_k$. Thus, our estimate $v_{k,i} \leq C_2 \cdot v_{k,i}^{(0)}$ from above, together with boundedness of β_1 , shows that we have

$$\begin{aligned} Y &\xrightarrow{\beta_1} Z|_{J \setminus J_B} \left(\left[\ell_{\theta}^{p_2^\nabla} (I_j) \right]_{j \in J \setminus J_B} \right) \\ &= X_{w^{(0)}} \left(\left[\ell_{\left(v_{k,i}^{(0)} / w_k^{(0)} \right)_i}^{p_2^\nabla} (I^{(k)}) \right]_{k \in J \setminus J_B} \right) \\ &= X \left(\left[\ell_{v^{(0)}}^{q_k^\nabla} (I^{(k)}) \right]_{k \in K} \right) \\ &\left(\text{since } v \leq C_2 \cdot v^{(0)} \right) \hookrightarrow X \left(\left[\ell_v^{q_k^\nabla} (I^{(k)}) \right]_{k \in K} \right), \end{aligned}$$

as required. The quantitative version of these considerations yields $\|\eta_2\| \leq C_2 \cdot \|\beta_1\|$.

All in all, Theorem 5.6 shows that ι_1 is well-defined and bounded and satisfies

$$\|\iota_1\| \leq C_3 \cdot \|\eta_1\| \|\eta_2\| \leq C_2 C_3 \cdot \|\beta_1\|$$

for some constant

$$C_3 = C_3(d, p_1, p_2, \mathcal{Q}, \mathcal{P}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}).$$

Note that $C_{\mathcal{P}, \Psi, p_2}$ can be estimated in terms of $C_{\mathcal{P}, \Psi, p_1}$ and d, p_1, \mathcal{P} , thanks to Corollary 5.4.

Boundedness of ι_2 : Here, we apply Theorem 5.6 with $K := I_0 := I$ and $J_0 := J_B$ and with

$$J^{(k)} := J_B \cap J_k = \{j \in J_B \mid P_j \cap Q_k \neq \emptyset\}, \quad \text{as well as} \quad I^{(k)} := k^{(2m+2)*} \quad \text{for } k \in K,$$

where $m := k(\mathcal{P}_{J_B}, \mathcal{Q})$. Furthermore, we select $q_k := p_1 \in [p_1, p_2]$ for all $k \in K$. This yields $q^{(0)} = \inf_{k \in K} q_k = p_1$. Note that Ψ is an L^{p_1} -BAPU for \mathcal{P} , as needed for application of Theorem 5.6.

Observe $J_0 = J_B \subset \bigcup_{k \in K} J^{(k)} = J_{00}$, as required in Theorem 5.6; indeed, for $j \in J_B$, there is some $\xi \in P_j \cap B \subset A \cup B = \mathcal{O} \cap \mathcal{O}'$. But since $\mathcal{Q} = (Q_i)_{i \in I}$ covers \mathcal{O} , there is some $k \in I$ satisfying $\xi \in Q_k$. Hence, $\xi \in P_j \cap Q_k \neq \emptyset$, which yields $j \in J_B \cap J_k = J^{(k)}$.

Now, let us show that condition (5.3) holds with our choices from above. Hence, assume towards a contradiction that there is some

$$\xi \in \left(\bigcup_{i \in I_0 \setminus I^{(k)}} Q_i \right) \cap \left(\bigcup_{j \in J^{(k)}} P_j \right) = \left(\bigcup_{i \in I \setminus k^{(2m+2)*}} Q_i \right) \cap \left(\bigcup_{j \in J_B \cap J_k} P_j \right).$$

Thus, there are $i \in I \setminus k^{(2m+2)*}$ and $j \in J_k \cap J_B$ with $\xi \in Q_i \cap P_j$. Because of $j \in J_B$, our assumptions imply $\xi \in P_j \subset Q_{i_j}^{m*}$ for some $i_j \in I$. But because of $\xi \in Q_i$, this implies $Q_i \cap Q_{i_j}^{m*} \neq \emptyset$ and thus $i_j \in i^{(m+1)*}$. Finally, since $j \in J_k$, we get $\emptyset \neq P_j \cap Q_k \subset Q_{i_j}^{m*} \cap Q_k$ and hence $k \in i_j^{(m+1)*} \subset i^{(2m+2)*}$, i.e., $i \in k^{(2m+2)*}$, in contradiction to $i \in I \setminus k^{(2m+2)*}$.

Next, we estimate the weights from equations (5.4) and (5.5). For the weight w from equation (5.4), we note for $k \in K$ and $j \in J^{(k)} = J_k \cap J_B$ (because of $q_k = p_1$) that

$$w_{k,j} = w_j^{(0)} := \begin{cases} |\det S_j|^{\frac{1}{p_2} - 1}, & \text{if } p_1 < 1, \\ |\det S_j|^{\frac{1}{p_2} - \frac{1}{p_1}}, & \text{if } p_1 \geq 1. \end{cases}$$

For the weight v from equation (5.5), we have to work slightly harder. First note for $k \in K$ and $i \in I^{(k)} = k^{(2m+2)*}$, as well as $j \in J^{(k)} = J_k \cap J_B$ that (by our assumptions) $P_j \subset Q_{i_j}^{m*}$ for some $i_j \in I$. Furthermore, there is some $\xi \in P_j \cap Q_k \subset Q_{i_j}^{m*} \cap Q_k$, so that $i_j \in k^{(m+1)*} \subset i^{(3m+3)*}$. Hence, $P_j \subset Q_{i_j}^{m*} \subset Q_i^{(4m+3)*}$. All in all, we get

$$\lambda(\overline{P_j} - \overline{Q_i}) \leq \lambda\left(\overline{Q_i^{(4m+3)*}} - \overline{Q_i^{(4m+3)*}}\right) \leq C_3 \cdot |\det T_i|,$$

for some constant $C_4 = C_4(d, m, \mathcal{Q}) \geq 1$ which is provided by Corollary 2.8. All in all, we derive (because of $q_k = p_1$) that

$$v_{k,i} = \begin{cases} [\sup_{j \in J^{(k)}} \lambda(\overline{P_j} - \overline{Q_i})]^{\frac{1}{p_1}-1} \leq C_3^{\frac{1}{p_1}-1} \cdot |\det T_i|^{\frac{1}{p_1}-1}, & \text{if } p_1 < 1, \\ 1, & \text{if } p_1 \geq 1. \end{cases}$$

We denote the right-hand side of this estimate by

$$v_i^{(0)} := \begin{cases} |\det T_i|^{\frac{1}{p_1}-1}, & \text{if } p_1 < 1, \\ 1, & \text{if } p_1 \geq 1, \end{cases}$$

so that we have shown $v_{k,i} \leq C_5 \cdot v_i^{(0)}$ for all $k \in K$, $i \in I^{(k)}$ and some constant $C_5 = C_5(d, p_1, m, \mathcal{Q})$. Note that $v^{(0)}$ is \mathcal{Q} -moderate, with $C_{v^{(0)}, \mathcal{Q}} \leq C_6 = C_6(d, p_1, \mathcal{Q})$, see equation (3.11). Hence, Lemma 4.13 shows that $X := Y_{1/v^{(0)}} \leq \mathbb{C}^I = \mathbb{C}^K$ is \mathcal{Q} -regular, with

$$\|\Gamma_{\mathcal{Q}}\|_{X \rightarrow X} \leq C_{v^{(0)}, \mathcal{Q}} \cdot \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y} \leq C_6 \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y} =: C_7.$$

It remains to verify boundedness of the embeddings η_1, η_2 from equations (5.6) and (5.7) of Theorem 5.6, with the preceding choice of $X = Y_{1/v^{(0)}}$. To this end, note that $Y \hookrightarrow X_{v^{(0)}}$ and that $X = Y_{1/v^{(0)}}$ is \mathcal{Q} -moderate, as we just saw. Hence, Lemma 5.10 yields

$$Y \hookrightarrow X \left(\left[\ell_{v^{(0)}}^{p_1^\nabla} (k^{(2m+2)*}) \right]_{k \in I} \right) \xrightarrow{(*)} X \left(\left[\ell_v^{q_k} (I^{(k)}) \right]_{k \in K} \right).$$

Here, the step marked with $(*)$ used $v_{k,i} \leq C_5 \cdot v_i^{(0)}$, as well as $I^{(k)} = k^{(2m+2)*}$ and $q_k = p_1$ for all $k \in K = I$. But this embedding precisely yields boundedness of η_2 from equation (5.7). One can easily make the preceding arguments quantitative (see Lemma 5.10), to derive $\|\eta_2\| \leq C_8$ for some constant $C_8 = C_8(d, p_1, m, \mathcal{Q}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y})$.

Finally, for the embedding η_1 from equation (5.6), we note for $k \in K$ and $j \in J^{(k)} = J_B \cap J_k$ that

$$\frac{w_{k,j}}{v_k^{(0)}} = \frac{w_j^{(0)}}{v_k^{(0)}} = \begin{cases} |\det S_j|^{\frac{1}{p_2}-1} / |\det T_k|^{\frac{1}{p_1}-1} = u_{k,j}, & \text{if } p_1 < 1, \\ |\det S_j|^{\frac{1}{p_2}-\frac{1}{p_1}} = u_{k,j}, & \text{if } p_1 \geq 1. \end{cases}$$

Thus, noting $J^{(k)} = J_B \cap J_k$ and recalling the definition of the (assumed) embedding β_2 from the statement of the present corollary, we derive

$$X \left(\left[\ell_w^{q_k^\Delta} (J^{(k)}) \right]_{k \in K} \right) = Y_{1/v^{(0)}} \left(\left[\ell_w^{p_1^\Delta} (J^{(k)}) \right]_{k \in K} \right) = Y \left(\left[\ell_u^{p_1^\Delta} (J_i \cap J_B) \right]_{i \in I} \right) \xrightarrow{\beta_2} Z|_{J_B},$$

which precisely yields boundedness of η_1 from equation (5.6), since we have $J_B = J_{00}$, as seen above. Quantitatively, we get $\|\eta_1\| \leq \|\beta_2\|$.

All in all, Theorem 5.6 shows that ι_2 as defined above is bounded; more precisely, Theorem 5.6 yields $\|\iota_2\| \leq C_9 \cdot \|\eta_1\| \|\eta_2\| \leq C_8 C_9 \cdot \|\beta_2\|$ for some constant

$$C_9 = C_9(d, p_1, p_2, \mathcal{Q}, \mathcal{P}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_1}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}).$$

Altogether, this yields the desired estimate

$$\|\iota\| \leq C_{10} \cdot (\|\iota_1\| + \|\iota_2\|) \lesssim \|\beta_1\| + \|\beta_2\|,$$

with an implied constant as in the statement of the corollary. Here, C_{10} is the triangle constant for $\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$, which can be bounded only in terms of p_2 and C_Z , see Theorem 3.21.

All that remains is to establish the final claim of the corollary, i.e., to estimate $\|\beta_1\|$ and $\|\beta_2\|$ in case of $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$.

For β_2 , which in this case reads

$$\beta_2 : \ell_w^{q_1} \left(\left[\ell_u^{p_1^\Delta} (J_i \cap J_B) \right]_{i \in I} \right) \hookrightarrow \ell_v^{q_2}(J_B),$$

we can simply use Corollary 5.12, with $r = p_1^\Delta$, $J_0 = J_B$ (since \mathcal{P}_{J_B} is almost subordinate to \mathcal{Q}) and with

$$u_i^{(1)} := \begin{cases} |\det T_i|^{1-\frac{1}{p_1}}, & \text{if } p_1 < 1, \\ 1 & \text{if } p_1 \geq 1, \end{cases} \quad \text{and} \quad u_j^{(2)} := \begin{cases} |\det S_j|^{p_2^{-1}-1}, & \text{if } p_1 < 1, \\ |\det S_j|^{p_2^{-1}-p_1^{-1}}, & \text{if } p_1 \geq 1. \end{cases}$$

Note that equation (3.11) shows that $u^{(1)}$ and $u^{(2)}$ are \mathcal{Q} -moderate and \mathcal{P} -moderate, respectively, with $C_{u^{(1)}, \mathcal{Q}} \leq C(d, p_1, \mathcal{Q})$ and $C_{u^{(2)}, \mathcal{P}} \leq C(d, p_1, p_2, \mathcal{P})$.

Hence, Corollary 5.12 yields

$$\|\beta_2\| \asymp \left\| \left(w_i^{-1} \cdot \left\| (v_j / u_{i,j})_{j \in J_B \cap J_i} \right\|_{\ell^{q_2 \cdot (p_1^\Delta / q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1 / q_2)'}},$$

where the implied constant only depends on $d, p_1, p_2, q_1, q_2, \mathcal{Q}, \mathcal{P}, k(\mathcal{P}_{J_B}, \mathcal{Q}), C_{w, \mathcal{Q}}$, as desired.

Finally, for β_1 , it is easy to see that boundedness of β_1 is equivalent to that of

$$\widetilde{\beta}_1 : \ell_w^{q_1}(L) \hookrightarrow \ell_v^{q_2} \left(\left[\ell_\theta^{p_2^\nabla}(I_j) \right]_{j \in J \setminus J_B} \right),$$

with $\|\beta_1\| = \|\widetilde{\beta}_1\|$. Thus, we use Corollary 5.8, with $r = p_2^\nabla$, $I_0 = L$ and $J_0 = J \setminus J_B$, as well as $u = \theta$ to obtain (because of $I_j \subset L$ and hence $I_0 \cap I_j = L \cap I_j = I_j$ for $j \in J \setminus J_B$) that

$$\|\widetilde{\beta}_1\| \asymp \left\| \left(v_j \cdot \left\| (w_i^{-1} \cdot \theta_i)_{i \in I_j} \right\|_{\ell^{p_2^\nabla \cdot (q_1 / p_2^\nabla)'}} \right)_{j \in J \setminus J_B} \right\|_{\ell^{q_2 \cdot (q_1 / q_2)'}},$$

where the implied constant only depends on $p_1, q_1, q_2, \mathcal{P}, k(\mathcal{Q}_{I_A}, \mathcal{P}), C_{v, \mathcal{P}}$. Note that we indeed have $I_0 = L = \bigcup_{j \in J_0} I_j$, as needed for the application of Corollary 5.8. Furthermore, as seen above, we have $I_0 = L \subset I_A$, so that $\mathcal{Q}_L = \mathcal{Q}_{I_0}$ is indeed almost subordinate to \mathcal{P} , with $k(\mathcal{Q}_L, \mathcal{P}) \leq k(\mathcal{Q}_{I_A}, \mathcal{P})$. \square

6. NECESSARY CONDITIONS FOR EMBEDDINGS

In this section, we study sharpness of the sufficient criteria for the existence of embeddings between decomposition spaces that we developed before.

Roughly speaking, the setting considered in this section is as follows: We assume that an embedding of the form

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z) \tag{6.1}$$

is true, where the two decomposition spaces are built with respect to the ((tight) semi-structured) coverings $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ of \mathcal{O} and $\mathcal{P} = (S_j P'_j + c_j)_{j \in J}$ of \mathcal{O}' . We will then show that the existence of such an embedding necessarily implies $p_1 \leq p_2$ and also the existence of certain embeddings between discrete sequence spaces, similar (and often identical) to the sufficient conditions which we derived in the previous section.

The section consists of four subsections. In the first subsection, we impose no special restrictions on the relation between \mathcal{Q} and \mathcal{P} . Nevertheless, we will show that the condition $p_1 \leq p_2$ —which was required in *all* sufficient conditions from the previous section—is a necessary consequence of the boundedness of the embedding (6.1). Furthermore, in case of $p_1 = p_2$, we will see that necessarily $\|\delta_j\|_Z \lesssim \|\delta_i\|_Y$ for all $i \in I$ and $j \in J$ for which $Q_i^\circ \cap P_j^\circ \neq \emptyset$.

In Subsection 6.2, we employ these elementary necessary criteria to show that an *equality*

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$$

of two decomposition spaces is only possible if we have $p_1 = p_2$ and if the coverings \mathcal{Q}, \mathcal{P} are *weakly equivalent*. Note though that in general this only holds for $p_1 \neq 2$. For the case of weighted Lebesgue sequence spaces $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$, however, we will be able to extend this result: In this case, we necessarily have $q_1 = q_2$ and \mathcal{Q} and \mathcal{P} are also equivalent for $p_1 = 2$, as long as $q_1 \neq 2$. Finally, we remark that in addition to the elementary results from the previous subsection, the proof of the equivalence between \mathcal{Q}, \mathcal{P} also uses simplified forms of the arguments used in Subsection 6.3. Thus, Subsection 6.2 serves as a gentle introduction to the remainder of the section.

From Subsection 6.3 on, we always assume (essentially) that \mathcal{Q} is almost subordinate to \mathcal{P} (or vice versa). Under this assumption, we will show that boundedness of the embedding (6.1) (essentially) implies boundedness of the embedding

$$Y \hookrightarrow Z \left(\left[\ell^{p_2}_{|\det T_i|^{p_1^{-1}-p_2^{-1}}} (I_j) \right]_{j \in J} \right).$$

This necessary condition almost coincides with the sufficient condition from Corollary 5.7; the only difference is that the “inner norm” for the necessary condition is ℓ^{p_2} , while it is $\ell^{p_2^\vee}$ for the sufficient condition. At least for $p_2 \in (0, 2]$, we have $p_2^\vee = p_2$, so that both conditions coincide. Thus, for a certain range of exponents, we achieve a *complete characterization*. A similar statement also holds if \mathcal{P} is almost subordinate to \mathcal{Q} .

Finally, in Subsection 6.5, in addition to \mathcal{Q} being almost subordinate to \mathcal{P} , we require \mathcal{Q} to be *relatively moderate* with respect to \mathcal{P} and we only consider weighted Lebesgue sequence spaces as the “global components” for the decomposition spaces. Under these more restrictive assumptions, we will show that the sufficient conditions from Corollary 5.7 are also necessary, so that we obtain a *complete characterization* for all possible exponents p_2 . With suitable changes, the same holds for the conditions from Corollary 5.11, i.e. if \mathcal{P} is almost subordinate to \mathcal{Q} .

Before we properly begin our investigation of necessary criteria for the existence of embeddings, we briefly indicate our proof strategy. Assuming an embedding $\iota : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$, we will “test” this embedding using suitably crafted functions:

For simplicity, we will assume $p_1 = p_2 = p \in [1, \infty]$. Given any (finitely supported) sequence $(c_j)_{j \in J}$, where $\mathcal{P} = (P_j)_{j \in J}$, we consider functions of the form

$$f = \sum_{j \in J} c_j \gamma_j \quad \text{where} \quad \gamma_j \in C_c^\infty(P_j).$$

If $\Phi = (\varphi_i)_{i \in I}$ is an L^p -BAPU for \mathcal{Q} , then $\varphi_i \equiv 0$ outside of Q_i . In particular, $\varphi_i \gamma_j \equiv 0$ for $j \notin J_i = \{\ell \in J \mid P_\ell \cap Q_i \neq \emptyset\}$ and thus

$$\|\mathcal{F}^{-1}(\varphi_i \cdot f)\|_{L^p} = \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \sum_{j \in J_i} c_j \gamma_j \right) \right\|_{L^p} \lesssim \left\| \mathcal{F}^{-1} \left(\sum_{j \in J_i} c_j \gamma_j \right) \right\|_{L^p} =: d_i \quad \forall i \in I. \quad (6.2)$$

Here, the last step used that we have $\|\mathcal{F}^{-1} \varphi_i\|_{L^1} \lesssim 1$ for all $i \in I$, as part of the definition of an L^p -BAPU. Consequently, using boundedness of ι , we get

$$\|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z)} \lesssim \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} \lesssim \|(d_i)_{i \in I}\|_Y < \infty,$$

at least if we assume $\|(d_i)_{i \in I}\|_Y < \infty$. Thus, our next goal is to obtain a *lower* bound on $\|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z)}$.

To this end, we use that admissibility of the covering \mathcal{P} implies (cf. the “*disjointization lemma*”, Lemma 2.14) that there is a finite partition $J = \bigsqcup_{r=1}^{r_0} J^{(r)}$ such that no two different sets P_j from the same index set $J^{(r)}$ are neighbors (of degree 3), i.e., such that $P_j^* \cap P_\ell^* = \emptyset$ for all $j, \ell \in J^{(r)}$ with $j \neq \ell$. Thus, if we fix $r \in \underline{r_0}$ and assume $c_j = 0$ for all $j \in J \setminus J^{(r)}$, we get

$$f = \sum_{j \in J^{(r)}} c_j \gamma_j.$$

The advantage of this additional assumption on the coefficients $(c_j)_{j \in J}$ is—among other things—that the different summands have disjoint support, which prevents cancellations in the sum. As a consequence of this and the fact that the family $(\gamma_j)_{j \in J}$ is adapted to the covering \mathcal{P} , one can show (cf. Lemma 6.3) that

$$\|(d_i)_{i \in I}\|_Y \gtrsim \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z)} \gtrsim \left\| \left(\|\mathcal{F}^{-1}(c_j \gamma_j)\|_{L^p} \right)_{j \in J} \right\|_Z =: \|(|c_j| \cdot e_j)_{j \in J} \|_Z.$$

Now, we have reached a crucial point: The sequence $(e_j)_{j \in J}$ given by $e_j = \|\mathcal{F}^{-1} \gamma_j\|_{L^p}$ is *highly invariant* under certain transformations, for example under switching from γ_j to the modulated version $\tilde{\gamma}_j = M_{z_j} \gamma_j$, with arbitrary $z_j \in \mathbb{R}^d$. Note that this transformation retains the property $\tilde{\gamma}_j \in C_c^\infty(P_j)$. In contrast to e_j , however, the sequence $(d_i)_{i \in I}$ defined in equation (6.2) is (in general) *not* invariant

under these transformations. More precisely, even though it is hard (or even impossible) to evaluate d_i in general, we will see (cf. Corollary 6.6) that a suitable choice of the modulations z_j entails

$$d_i = \left\| \mathcal{F}^{-1} \left(\sum_{j \in J_i} c_j \cdot M_{z_j} \gamma_j \right) \right\|_{L^p} \asymp \left\| (|c_j| \cdot \|\mathcal{F}^{-1} \gamma_j\|_{L^p})_{j \in J_i} \right\|_{\ell^p} = \left\| (|c_j| \cdot e_j)_{j \in J_i} \right\|_{\ell^p}$$

Thus, in the spirit of [18], we found a possibility to *arbitrage* the embedding, i.e., to exploit a differing degree of symmetry between both sides of the embedding.

All in all, we arrive at

$$\left\| (|c_j| \cdot e_j)_{j \in J} \right\|_Z \lesssim \|(d_i)_{i \in I}\|_Y \asymp \left\| \left(\left\| (|c_j| \cdot e_j)_{j \in J_i} \right\|_{\ell^p} \right)_{i \in I} \right\|_Y,$$

for (almost) arbitrary sequences $(c_j)_{j \in J}$ supported in $J^{(r)}$. Since $r \in \underline{r_0}$ was arbitrary, this restriction on the support can easily be removed. Careful readers will notice that the argument sketched here is used (in a slightly more precise, general and elaborate form) in the proof of Theorem 6.13.

Now that we have illustrated the general idea, we are ready to properly start our investigations.

6.1. Elementary necessary conditions. In the following, we will actually consider a slightly more general setting than in equation (6.1). Precisely, we will assume that there is an embedding

$$\left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)} \right) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z), \quad (6.3)$$

for some subset $K \subset \mathcal{O} \cap \mathcal{O}'$, where

$$\mathcal{D}_K := \mathcal{D}_K^{\mathcal{Q}, p_1, Y} := \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \cap \{f \in C_c^\infty(\mathbb{R}^d) \mid \text{supp } f \subset K\}$$

are all those test functions with support in K which also belong to the Fourier-side decomposition spaces $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$. As long as there is no chance for confusion, we will simply write \mathcal{D}_K instead of $\mathcal{D}_K^{\mathcal{Q}, p_1, Y}$. Finally, if $\delta_i \in Y$ for all $i \in I$, it is not hard to see $C_c^\infty(\mathcal{O}) \subset \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$, which implies $\mathcal{D}_K^{\mathcal{Q}, p_1, Y} = \{f \in C_c^\infty(\mathbb{R}^d) \mid \text{supp } f \subset K\}$. A similar argument shows that $\mathcal{D}_K^{\mathcal{Q}, p_1, Y}$ is independent of the choice of p_1 , but we will not use this in the following.

Our first necessary condition is rather simple: We observe that in Theorem 5.6 and in Corollaries 5.7, 5.11 and 5.14, the inequality $p_1 \leq p_2$ was always part of the assumptions. Our first necessary condition will show that this is inevitable. It is worth noting that we do not need to impose any additional assumptions (like subordinateness) on the coverings \mathcal{Q}, \mathcal{P} .

Lemma 6.1. *Let $\emptyset \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p_1, p_2 \in (0, \infty]$ and assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an L^{p_1} -decomposition covering of \mathcal{O} and that $\mathcal{P} = (P_j)_{j \in J}$ is an L^{p_2} -decomposition covering of \mathcal{O}' . Let $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ be \mathcal{Q} -regular and \mathcal{P} -regular, respectively.*

Finally, let $K \subset \mathbb{R}^d$ and assume that there are $i \in I$ and $j \in J$ with $K^\circ \cap Q_i^\circ \cap P_j^\circ \neq \emptyset$ and with $\delta_i \in Y$. If the map

$$\iota : \left(\mathcal{D}_K^{\mathcal{Q}, p_1, Y}, \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)} \right) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z), f \mapsto f$$

is well-defined and continuous, then we have $p_1 \leq p_2$.

In case of $p_1 = p_2$, there exists a constant $C > 0$ depending only on

$$d, p_1, p_2, \mathcal{Q}, \mathcal{P}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}$$

such that $\delta_j \in Z$ and $\|\delta_j\|_Z \leq C \|\iota\| \cdot \|\delta_i\|_Y$ holds for all $(i, j) \in I \times J$ with $K^\circ \cap Q_i^\circ \cap P_j^\circ \neq \emptyset$ and with $\delta_i \in Y$.

Here, the L^{p_1} -BAPU Φ and the L^{p_2} -BAPU Ψ have to be used to compute the (quasi)-norms on $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$ and $\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$, respectively. \blacktriangleleft

Proof. Let $\Phi = (\varphi_i)_{i \in I}$ be an L^{p_1} -BAPU for \mathcal{Q} and let $\Psi = (\psi_j)_{j \in J}$ be an L^{p_2} -BAPU for \mathcal{P} . Fix a nontrivial test function $\theta \in C_c^\infty(B_1(0))$. We will prove both parts of the lemma (more or less) simultaneously. The assumptions yield $\xi_0 \in \mathbb{R}^d$ and $\varepsilon > 0$ with $B_\varepsilon(\xi_0) \subset K^\circ \cap Q_i^\circ \cap P_j^\circ$.

For $n \in \mathbb{N}$, define

$$\theta_n : \mathbb{R}^d \rightarrow \mathbb{C}, \xi \mapsto \theta\left(\frac{n}{\varepsilon}(\xi - \xi_0)\right),$$

i.e. $\theta_n = L_{\xi_0} [\Delta_{n/\varepsilon} \theta]$. We then have

$$\theta_n \in C_c^\infty(B_{\varepsilon/n}(\xi_0)) \subset C_c^\infty(B_\varepsilon(\xi_0)) \subset C_c^\infty(K^\circ \cap P_j^\circ \cap Q_i^\circ).$$

But Lemma 2.4 implies $\psi_j^* \equiv 1$ on $P_j \supset \text{supp } \theta_n$, and hence $\psi_j^* \theta_n = \theta_n$.

Now note that $\varphi_\ell \theta_n \neq 0$ yields some $\xi \in \text{supp } \theta_n \subset Q_i$ with $\varphi_\ell(\xi) \neq 0$. Hence, $\xi \in Q_\ell \cap Q_i \neq \emptyset$, i.e. $\ell \in i^*$. But for $\ell \in i^*$, there are two cases: For $p_1 \in [1, \infty]$, Young's inequality ($L^1 * L^{p_1} \hookrightarrow L^{p_1}$) yields

$$\begin{aligned} \|\mathcal{F}^{-1}(\varphi_\ell \theta_n)\|_{L^{p_1}} &= \|(\mathcal{F}^{-1}\varphi_\ell) * (\mathcal{F}^{-1}\theta_n)\|_{L^{p_1}} \\ &\leq \|\mathcal{F}^{-1}\varphi_\ell\|_{L^1} \cdot \|\mathcal{F}^{-1}\theta_n\|_{L^{p_1}} \\ &\leq C_{\mathcal{Q}, \Phi, p_1} \cdot \|\mathcal{F}^{-1}\theta_n\|_{L^{p_1}} < \infty. \end{aligned}$$

Otherwise, in case of $p_1 \in (0, 1)$, we know (by definition of an L^{p_1} -decomposition covering) that $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ is a semi-structured covering. Furthermore, we note $\text{supp } \varphi_\ell \subset \overline{Q_\ell} \subset \overline{Q_\ell}^*$ and $\text{supp } \theta_n \subset Q_i^\circ \subset \overline{Q_\ell}^*$, so that Corollary 3.14 yields a constant $C_1 = C_1(\mathcal{Q}, d, p_1) > 0$ with

$$\begin{aligned} \|\mathcal{F}^{-1}(\varphi_\ell \theta_n)\|_{L^{p_1}} &\leq C_1 \cdot |\det T_\ell|^{\frac{1}{p_1}-1} \cdot \|\mathcal{F}^{-1}\varphi_\ell\|_{L^{p_1}} \|\mathcal{F}^{-1}\theta_n\|_{L^{p_1}} \\ &\leq C_1 C_{\mathcal{Q}, \Phi, p_1} \cdot \|\mathcal{F}^{-1}\theta_n\|_{L^{p_1}} \end{aligned}$$

for all $\ell \in i^*$.

If we set $C_1 := 1$ for $p_1 \in [1, \infty]$, then a combination of the two cases from easily yields

$$\|\mathcal{F}^{-1}(\varphi_\ell \theta_n)\|_{L^{p_1}} \leq C_1 C_{\mathcal{Q}, \Phi, p_1} \cdot \|\mathcal{F}^{-1}\theta_n\|_{L^{p_1}} \cdot \mathbb{1}_{i^*}(\ell) \quad \forall \ell \in I.$$

But since Y is a \mathcal{Q} -regular—in particular solid—sequence space with $\delta_i \in Y$ and since $\mathbb{1}_{i^*} = \Gamma_{\mathcal{Q}} \delta_i$, we get $\theta_n \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$ (and hence $\theta_n \in \mathcal{D}_K^{\mathcal{Q}, p_1, Y}$) with

$$\begin{aligned} \|\theta_n\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)} &= \left\| \left(\|\mathcal{F}^{-1}(\varphi_\ell \theta_n)\|_{L^{p_1}} \right)_{\ell \in I} \right\|_Y \\ &\leq C_1 C_{\mathcal{Q}, \Phi, p_1} \cdot \|\mathcal{F}^{-1}\theta_n\|_{L^{p_1}} \cdot \|\mathbb{1}_{i^*}\|_Y \\ &\leq C_1 C_{\mathcal{Q}, \Phi, p_1} \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y} \cdot \|\delta_i\|_Y \cdot \|\mathcal{F}^{-1}\theta_n\|_{L^{p_1}} \\ &=: C_2 \cdot \|\delta_i\|_Y \cdot \|\mathcal{F}^{-1}\theta_n\|_{L^{p_1}} < \infty \end{aligned}$$

for all $n \in \mathbb{N}$.

Since we assume ι to be well-defined and bounded, we get $\theta_n \in \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$ and

$$\begin{aligned} \|\theta_n\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)} &\leq \|\iota\| \cdot \|\theta_n\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)} \\ &\leq C_2 \|\iota\| \cdot \|\delta_i\|_Y \cdot \|\mathcal{F}^{-1}\theta_n\|_{L^{p_1}}. \end{aligned}$$

But Remark 3.16 shows that the family $\Gamma := (\gamma_\ell)_{\ell \in J} := (\psi_\ell^*)_{\ell \in J}$ is an L^{p_2} -bounded family for \mathcal{P} , with $C_{\mathcal{P}, \Gamma, p_2} \leq C(\mathcal{P}, C_{\mathcal{P}, \Psi, p_2}, d, p_2)$, so that Theorem 3.17 yields $C_3 = C_3(\mathcal{P}, p_2, d, C_{\mathcal{P}, \Gamma, p_2}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}) > 0$ with

$$\begin{aligned} \|\theta_n\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)} &\geq C_3^{-1} \cdot \left\| \left(\|\mathcal{F}^{-1}(\psi_\ell^* \theta_n)\|_{L^{p_2}} \right)_{\ell \in J} \right\|_Z \\ (Z \text{ solid and } \psi_j^* \theta_n = \theta_n) &\geq C_3^{-1} \cdot \|\mathcal{F}^{-1}\theta_n\|_{L^{p_2}} \cdot \|\delta_j\|_Z. \end{aligned}$$

In particular, $\delta_j \in Z$.

Altogether, we finally arrive at

$$\|\mathcal{F}^{-1}\theta_n\|_{L^{p_2}} \cdot \|\delta_j\|_Z \leq C_2 C_3 \|\iota\| \cdot \|\delta_i\|_Y \cdot \|\mathcal{F}^{-1}\theta_n\|_{L^{p_1}} \quad (6.4)$$

for all $n \in \mathbb{N}$. Furthermore, we can estimate

$$C_2 C_3 \leq C_4 = C_4(d, p_1, p_2, \mathcal{Q}, \mathcal{P}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}).$$

Now, note $\mathcal{F}^{-1}\theta_n = \left(\frac{\varepsilon}{n}\right)^d \cdot e^{2\pi i \langle \cdot, \xi_0 \rangle} \cdot (\mathcal{F}^{-1}\theta) \left(\frac{\varepsilon}{n} \cdot \right)$ and hence

$$\begin{aligned} \|\mathcal{F}^{-1}\theta_n\|_{L^p} &= \left(\frac{\varepsilon}{n}\right)^d \cdot \|\Delta_{\frac{\varepsilon}{n}}(\mathcal{F}^{-1}\theta)\|_{L^p} \\ &= \left(\frac{\varepsilon}{n}\right)^{d(1-\frac{1}{p})} \cdot \|\mathcal{F}^{-1}\theta\|_{L^p} \end{aligned}$$

for all $p \in (0, \infty]$. Thus, estimate (6.4) yields

$$\|\mathcal{F}^{-1}\theta\|_{L^{p_2}} \cdot \left(\frac{\varepsilon}{n}\right)^{d\left(1-\frac{1}{p_2}\right)} \cdot \|\delta_j\|_Z \leq C_4 \|\iota\| \cdot \|\delta_i\|_Y \cdot \left(\frac{\varepsilon}{n}\right)^{d\left(1-\frac{1}{p_1}\right)} \cdot \|\mathcal{F}^{-1}\theta\|_{L^{p_1}}$$

and hence

$$n^{\frac{1}{p_2}-\frac{1}{p_1}} \leq C_4 \|\iota\| \varepsilon^{d\left(\frac{1}{p_2}-\frac{1}{p_1}\right)} \cdot \frac{\|\delta_i\|_Y}{\|\delta_j\|_Z} \cdot \frac{\|\mathcal{F}^{-1}\theta\|_{L^{p_1}}}{\|\mathcal{F}^{-1}\theta\|_{L^{p_2}}} \quad \forall n \in \mathbb{N},$$

which can only hold if $\frac{1}{p_2} - \frac{1}{p_1} \leq 0$, i.e. if $p_1 \leq p_2$. This yields the first claim.

Finally, in case of $p_1 = p_2$, most terms cancel, so that we get $\|\delta_j\|_Z \leq C_4 \|\iota\| \cdot \|\delta_i\|_Y$ with C_4 independent of i, j . \square

6.2. Coincidence of decomposition spaces. In this subsection, we assume that we are given two coverings \mathcal{Q}, \mathcal{P} of a common set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ such that we have

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2}) \quad (6.5)$$

for certain $p_1, p_2, q_1, q_2 \in (0, \infty]$ and certain weights w, v which are \mathcal{Q} -moderate and \mathcal{P} -moderate, respectively. Given these assumptions, we will be able to show that the “ingredients” for the two decomposition spaces are essentially the same. Precisely, we will show $p_1 = p_2$, $q_1 = q_2$ and $w_i \asymp v_j$ given $Q_i \cap P_j \neq \emptyset$. Finally, in case of $(p_1, q_1) \neq (2, 2)$, we will also show that the two coverings \mathcal{Q}, \mathcal{P} are **weakly equivalent**. Recall from Definition 2.10 that this means that the sets

$$I_j := \{i \in I \mid Q_i \cap P_j \neq \emptyset\} \quad \text{and} \quad J_i := \{j \in J \mid P_j \cap Q_i \neq \emptyset\}$$

satisfy $|I_j| \leq C$ and $|J_i| \leq C$ for all $i \in I$ and $j \in J$. As the final result of this section, we show that these properties together are sufficient for the equality (6.5) to hold, at least for $p_1 \in [1, \infty]$. In the Quasi-Banach regime $p_1 \in (0, 1)$, we will have to impose certain additional technical assumptions.

We thus obtain an (almost) complete characterization of the equality of two decomposition spaces in terms of their “ingredients”. Note though that in order to get the full characterization as explained above, we need to restrict ourselves to global components of the form $Y = \ell_w^{q_1}$ instead of general (\mathcal{Q}) -regular sequence spaces. For general (\mathcal{Q}) -regular sequence spaces, we will nevertheless be able to show that $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$ can only hold if $p_1 = p_2$; in case of $p_1 \neq 2$, a further necessary condition is that \mathcal{Q} and \mathcal{P} are weakly equivalent.

Finally, the methods of proof in this subsection provide a nice preparation for the following subsections, which are slightly more involved. One general technique which we will use again and again is to “test” an embedding between (or an equality of) two Fourier-side decomposition spaces by considering functions of the form

$$f_{z,c,\varepsilon} = \sum_{i \in I} M_{z_i}(\varepsilon_i c_i \cdot \gamma_i) \quad (6.6)$$

for certain functions γ_i which are supported (essentially) in Q_i and for (more or less) arbitrary sequences $\varepsilon = (\varepsilon_i)_{i \in I} \in \{\pm 1\}^I$ and $c = (c_i)_{i \in I} \in \ell_0(I)$. With this choice, the (quasi)-norm $\|f_{z,c,\varepsilon}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)}$ is (essentially) independent of the choice of the modulations $z_i \in \mathbb{R}^d$ and of the signs ε_i , since modulation does not change the support (on the Fourier side) of the individual summands and since the individual summands are (essentially) disjointly supported. Note that we are considering the quasi-norm with respect to \mathcal{Q} , i.e. with respect to the covering to which the γ_i are adapted.

In general, this independence from the choice of $z = (z_i)_{i \in I}$ and $\varepsilon = (\varepsilon_i)_{i \in I}$ does *not* hold for the (quasi)-norm $\|f_{z,c,\varepsilon}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)}$. As we will see (cf. Corollary 6.6), a suitable choice of z often makes it possible to derive a certain embedding between sequence spaces as a result of the embedding for decomposition spaces. A similar statement holds for a suitable choice of ε . In the spirit of [18], we have thus found a possibility to *arbitrage* the embedding, cf. also the introduction to this section.

Our next lemma provides one of two estimates for the (essential) independence of $\|f_{z,c,\varepsilon}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)}$ from the choice of z .

Lemma 6.2. *Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open, let $p \in (0, \infty]$ and assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an L^p -decomposition covering of \mathcal{O} with L^p -BAPU $\Phi = (\varphi_i)_{i \in I}$. Finally, let $Y \leq \mathbb{C}^I$ be \mathcal{Q} -regular.*

For each $k \in \mathbb{N}_0$, there is a constant

$$C = C(k, d, p, \mathcal{Q}, C_{\mathcal{Q}, \Phi, p}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}) > 0$$

such that for arbitrary coefficients $(c_i)_{i \in I} \in \ell_0(I)$, functions $(\gamma_i)_{i \in I}$ with $\gamma_i \in C_c^\infty(\mathcal{O})$ and $\gamma_i \equiv 0$ on $(Q_i^{k*})^c$, signs $\varepsilon = (\varepsilon_i)_{i \in I} \in \{\pm 1\}^I$ and modulations $(z_i)_{i \in I} \in (\mathbb{R}^d)^I$, the estimate

$$\left\| \sum_{i \in I} M_{z_i}(\varepsilon_i c_i \cdot \gamma_i) \right\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)} \leq C \cdot \left\| (c_i \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p})_{i \in I} \right\|_Y \quad (6.7)$$

holds, provided that $(c_i \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p})_{i \in I} \in Y$. In particular, $\sum_{i \in I} M_{z_i}(\varepsilon_i c_i \cdot \gamma_i) \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$. ◀

Remark. Since the sequence $(c_i)_{i \in I} \in \ell_0(I)$ is finitely supported and since $\gamma_i \in C_c^\infty(\mathcal{O}) \subset \mathcal{S}(\mathbb{R}^d)$, we have $(c_i \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p})_{i \in I} \in Y$ as soon as $\delta_i \in Y$ for all $i \in I$, which holds for essentially all reasonable “global components” Y , in particular for $Y = \ell_w^q(I)$. ♦

Proof. Let $\ell \in I$ be arbitrary. For $i \in I$, we note that

$$0 \neq \varphi_\ell \cdot M_{z_i}(\varepsilon_i c_i \cdot \gamma_i) = \varepsilon_i c_i \cdot M_{z_i}(\varphi_\ell \cdot \gamma_i)$$

implies $\emptyset \neq Q_\ell \cap Q_i^{k*}$ and thus $i \in \ell^{(k+1)*}$.

As usual, we distinguish the two cases $p \in [1, \infty]$ and $p \in (0, 1)$. For $p \in [1, \infty]$, Young’s inequality $(L^1 * L^p \hookrightarrow L^p)$, together with the identity $\mathcal{F}^{-1}[M_z g] = L_{-z}[\mathcal{F}^{-1}g]$ and with translation invariance of $\|\cdot\|_{L^p}$, implies

$$\begin{aligned} \|\mathcal{F}^{-1}[M_{z_i}(\varphi_\ell \cdot \gamma_i)]\|_{L^p} &= \|\mathcal{F}^{-1}[\varphi_\ell \cdot \gamma_i]\|_{L^p} \\ &\leq \|\mathcal{F}^{-1} \varphi_\ell\|_{L^1} \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p} \\ &\leq C_{\mathcal{Q}, \Phi, p} \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p}. \end{aligned}$$

In case of $p \in (0, 1)$, the definition of an L^p -decomposition covering shows that $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ is semi-structured. Next, we observe $\text{supp } \varphi_\ell \subset \overline{Q_\ell} \subset \overline{Q_\ell^{(2k+1)*}}$ and $\text{supp } \gamma_i \subset \overline{Q_i^{k*}} \subset \overline{Q_\ell^{(2k+1)*}}$, because of $i \in \ell^{(k+1)*}$. Hence, Corollary 3.14 yields a constant $C_1 = C_1(\mathcal{Q}, k, d, p)$ with

$$\begin{aligned} \|\mathcal{F}^{-1}[M_{z_i}(\varphi_\ell \cdot \gamma_i)]\|_{L^p} &= \|\mathcal{F}^{-1}[\varphi_\ell \cdot \gamma_i]\|_{L^p} \\ &\leq C_1 \cdot |\det T_\ell|^{\frac{1}{p}-1} \|\mathcal{F}^{-1} \varphi_\ell\|_{L^p} \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p} \\ &\leq C_1 C_{\mathcal{Q}, \Phi, p} \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p}. \end{aligned}$$

Thus, if we set $C_1 := 1$ for $p \in [1, \infty]$, this estimate is valid for all $p \in (0, \infty]$.

Since $\|\cdot\|_{L^p}$ is a quasi-norm (with triangle constant only depending on p) and because of the uniform bound $|\ell^{(k+1)*}| \leq N_{\mathcal{Q}}^{k+1}$, which was established in Lemma 2.9, there is a constant $C_2 = C_2(p, k, \mathcal{Q})$ satisfying

$$\begin{aligned} \left\| \mathcal{F}^{-1} \left[\varphi_\ell \cdot \sum_{i \in I} M_{z_i}(\varepsilon_i c_i \cdot \gamma_i) \right] \right\|_{L^p} &\stackrel{(\dagger)}{\leq} C_2 \sum_{i \in \ell^{(k+1)*}} [|c_i| \cdot \|\mathcal{F}^{-1}[M_{z_i}(\varphi_\ell \cdot \gamma_i)]\|_{L^p}] \\ &\leq C_1 C_2 C_{\mathcal{Q}, \Phi, p} \cdot \sum_{i \in \ell^{(k+1)*}} [|c_i| \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p}] \\ &= C_1 C_2 C_{\mathcal{Q}, \Phi, p} \cdot (\Theta_{k+1} \zeta)_\ell. \end{aligned}$$

In the last step, we introduced the sequence $\zeta = (\zeta_i)_{i \in I}$ defined by $\zeta_i := |c_i| \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p}$ for $i \in I$. Furthermore, we used the $(k+1)$ -fold **clustering map** $\Theta_{k+1} : Y \rightarrow Y$ as defined in Lemma 3.9. At (\dagger) , we used that $\varphi_\ell \cdot M_{z_i}(\varepsilon_i c_i \cdot \gamma_i) \neq 0$ can only hold for $i \in \ell^{(k+1)*}$, as observed at the beginning of the proof.

Let $f := \sum_{i \in I} M_{z_i}(\varepsilon_i c_i \cdot \gamma_i)$. Note that our assumptions and the solidity of Y imply $\zeta \in Y$ and thus $\Theta_{k+1} \zeta \in Y$. But the estimate above showed $\|\mathcal{F}^{-1}(\varphi_\ell \cdot f)\|_{L^p} \lesssim (\Theta_{k+1} \zeta)_\ell$ for all $\ell \in I$. Hence, the solidity of Y yields $(\|\mathcal{F}^{-1}(\varphi_\ell \cdot f)\|_{L^p})_{\ell \in I} \in Y$ and thus $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$. To obtain a quantitative estimate, note that Lemma 3.9 yields $\|\Theta_{k+1}\| \leq C_3$ for some constant $C_3 = C_3(k, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y})$. Finally,

define $C_4 := C_1 C_2 C_3 C_{\mathcal{Q}, \Phi, p}$ and conclude (using the solidity of Y) that

$$\begin{aligned} \left\| \sum_{i \in I} M_{z_i} (\varepsilon_i c_i \cdot \gamma_i) \right\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)} &\leq C_1 C_2 C_{\mathcal{Q}, \Phi, p} \cdot \|\Theta_{k+1} \zeta\|_Y \\ &\leq C_4 \cdot \|(|c_i| \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p})_{i \in I}\|_Y \\ &= C_4 \cdot \|(c_i \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p})_{i \in I}\|_Y. \end{aligned}$$

This completes the proof. \square

The reverse of the previous estimate is not true in general, since there could be cancellations between neighboring indices, e.g. $\gamma_i + \gamma_\ell = 0$ for certain $i \neq \ell$. We will see in the next lemma, however, that this can be excluded by introducing suitable assumptions:

Lemma 6.3. *Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open, let $p \in (0, \infty]$ and assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an L^p -decomposition covering of \mathcal{O} . Finally, let $Y \leq \mathbb{C}^I$ be \mathcal{Q} -regular.*

For each $k \in \mathbb{N}_0$, there is a constant

$$C = C(\mathcal{Q}, p, d, k, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}) > 0$$

such that the following holds: If

- $I_0 \subset I$ with $Q_i^{k*} \cap Q_\ell^{k*} = \emptyset$ for all $i, \ell \in I_0$ with $i \neq \ell$,
- $(c_i)_{i \in I_0} \in \ell_0(I_0)$,
- $(\phi_i)_{i \in I}$ is an L^p -bounded system for \mathcal{Q} with $\phi_i \equiv 0$ on $\mathcal{O} \setminus Q_i^{k*}$ for all $i \in I$,
- $(\gamma_i)_{i \in I_0}$ with $\gamma_i \in C_c^\infty(\mathcal{O})$ and $\phi_i \gamma_i = \gamma_i$ for all $i \in I_0$,
- $\varepsilon = (\varepsilon_i)_{i \in I_0} \in \{\pm 1\}^{I_0}$ and
- $(z_i)_{i \in I_0} \in (\mathbb{R}^d)^{I_0}$,
- $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ with $\phi_i f = \phi_i g$ for all $i \in I_0$, where $g := \sum_{i \in I_0} M_{z_i} (\varepsilon_i c_i \cdot \gamma_i)$,

then

$$\left\| (c_i \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p})_{i \in I_0} \right\|_{Y|_{I_0}} \leq C \cdot C_{\mathcal{Q}, (\phi_i)_{i \in I}, p} \cdot \|f\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)}, \quad (6.8)$$

In particular, $(c_i \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p})_{i \in I_0} \in Y|_{I_0}$. \blacktriangleleft

Proof. Our assumption on the L^p -bounded system $(\phi_i)_{i \in I}$ shows that we can choose $\ell_{(\phi_i)_{i \in I}, \mathcal{Q}} = k$. Therefore, Theorem 3.17 (with the partition $(I^{(r)})_{r=1}$ of I consisting of the single element $I^{(1)} := I$ and with the L^p -bounded system $(\phi_i)_{i \in I}$) yields a constant

$$C_1 = C_1(\mathcal{Q}, p, d, k, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}) > 0$$

with

$$\begin{aligned} C_1 C_{\mathcal{Q}, (\phi_i)_{i \in I}, p} \cdot \|f\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)} &\geq \|f\|_{(\phi_i)_{i \in I}, (I^{(r)})_{r=1}, L^p, Y} \\ &= \left\| (\|\mathcal{F}^{-1}(\phi_i \cdot f)\|_{L^p})_{i \in I} \right\|_Y = \left\| \zeta^{(f)} \right\|_Y \end{aligned}$$

for all $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$, where we defined $\zeta^{(f)}$ by $\zeta_i^{(f)} := \|\mathcal{F}^{-1}(\phi_i \cdot f)\|_{L^p}$. In particular, we have $\zeta^{(f)} \in Y$ for all $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$.

Now, we will apply this to the given function f from the assumption. To this end, let $\ell \in I_0$ be arbitrary. For $i \in I_0$, there are two cases:

- Case 1.* We have $i \neq \ell$. By our assumption on I_0 , this implies $Q_i^{k*} \cap Q_\ell^{k*} = \emptyset$. But because of $\phi_i \equiv 0$ on $\mathcal{O} \setminus Q_i^{k*}$, this yields $\phi_i \phi_\ell \equiv 0$. Using $\gamma_i = \phi_i \gamma_i$, we finally get $\phi_\ell \gamma_i = \phi_\ell \phi_i \gamma_i \equiv 0$.
- Case 2.* We have $i = \ell$. In this case, our assumptions yield $\phi_\ell \gamma_i = \phi_i \gamma_i = \gamma_i$.

All in all, these considerations yield—together with our assumption $\phi_i f = \phi_i g$ for all $i \in I_0$ —that

$$\begin{aligned} \phi_\ell f &= \phi_\ell g = \phi_\ell \cdot \sum_{i \in I_0} M_{z_i} (\varepsilon_i c_i \cdot \gamma_i) \\ &= \sum_{i \in I_0} M_{z_i} (\varepsilon_i c_i \cdot \phi_\ell \gamma_i) \\ &= M_{z_\ell} (\varepsilon_\ell c_\ell \cdot \gamma_\ell) \end{aligned}$$

for arbitrary $\ell \in I_0$. But this means

$$\begin{aligned} \zeta_\ell^{(f)} &= \|\mathcal{F}^{-1}(\phi_\ell \cdot f)\|_{L^p} \\ &= \|\mathcal{F}^{-1}[M_{z_\ell}(\varepsilon_\ell c_\ell \cdot \gamma_\ell)]\|_{L^p} \\ &= |c_\ell| \cdot \|L_{-z_\ell}[\mathcal{F}^{-1}\gamma_\ell]\|_{L^p} \\ &= |c_\ell| \cdot \|\mathcal{F}^{-1}\gamma_\ell\|_{L^p} =: \omega_\ell \end{aligned}$$

for all $\ell \in I_0$. If we set $\omega_\ell := 0$ for $\ell \in I \setminus I_0$, we have thus shown $0 \leq \omega_\ell \leq \zeta_\ell^{(f)}$ for all $\ell \in I$. Using solidity of Y , this implies $\omega := (\omega_\ell)_{\ell \in I} \in Y$ with

$$\left\| (c_i \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p})_{i \in I_0} \right\|_{Y|_{I_0}} = \|\omega\|_Y \leq \|\zeta^{(f)}\|_Y \leq C_1 C_{\mathcal{Q}, (\phi_i)_{i \in I}, p} \cdot \|f\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)},$$

as desired. \square

Occasionally, the following specialized version of the preceding lemma is more convenient to use.

Corollary 6.4. *Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open, let $p \in (0, \infty]$ and assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an L^p -decomposition covering of \mathcal{O} with L^p -BAPU $\Phi = (\varphi_i)_{i \in I}$. Finally, let $Y \leq \mathbb{C}^I$ be \mathcal{Q} -regular.*

For each $k \in \mathbb{N}_0$, there is a constant

$$C = C(\mathcal{Q}, p, d, k, C_{\mathcal{Q}, \Phi, p}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}) > 0$$

such that the following holds: If

- $I_0 \subset I$ with $Q_i^{(k+1)*} \cap Q_\ell^{(k+1)*} = \emptyset$ for all $i, \ell \in I_0$ with $i \neq \ell$,
- $(c_i)_{i \in I_0} \in \ell_0(I_0)$,
- $(\gamma_i)_{i \in I_0}$ with $\gamma_i \in C_c^\infty(\mathcal{O})$ and $\gamma_i \equiv 0$ on $\mathcal{O} \setminus Q_i^{k*}$ for all $i \in I_0$,
- $\varepsilon = (\varepsilon_i)_{i \in I_0} \in \{\pm 1\}^{I_0}$ and
- $(z_i)_{i \in I_0} \in (\mathbb{R}^d)^{I_0}$,

then

$$\left\| \sum_{i \in I_0} M_{z_i} (\varepsilon_i c_i \cdot \gamma_i) \right\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)} \geq C^{-1} \cdot \left\| (c_i \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p})_{i \in I_0} \right\|_{Y|_{I_0}}, \quad (6.9)$$

provided that $\sum_{i \in I_0} M_{z_i} (\varepsilon_i c_i \cdot \gamma_i) \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$. In particular, $(c_i \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p})_{i \in I_0} \in Y|_{I_0}$. \blacktriangleleft

Remark. For brevity, let $f := \sum_{i \in I_0} M_{z_i} (\varepsilon_i c_i \cdot \gamma_i)$. The lemma shows for $\zeta := (c_i \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p})_{i \in I_0}$ that $\zeta \in Y|_{I_0}$ is a necessary condition for $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$. Furthermore, it yields a corresponding (quasi)-norm estimate.

Conversely, an application of Lemma 6.2 (to the extended sequences $(c_i)_{i \in I}$ and $(\gamma_i)_{i \in I}$ with $c_i = 0$ and $\gamma_i \equiv 0$ for $i \in I \setminus I_0$) shows that we have $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ as soon as $(c_i \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p})_{i \in I} \in Y$, i.e. as soon as $\zeta \in Y|_{I_0}$. Furthermore, in this case, the same lemma shows $\|f\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)} \lesssim \|\zeta\|_{Y|_{I_0}}$, where the constant depends on the same quantities as in Lemma 6.3 (and on $C_{\mathcal{Q}, \Phi, p}$).

In summary, a combination of both lemmata shows $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y) \iff \zeta \in Y|_{I_0}$ and also

$$\|f\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)} \asymp \left\| (c_i \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p})_{i \in I_0} \right\|_{Y|_{I_0}}, \quad (6.10)$$

where the implied constant only depends on $\mathcal{Q}, p, d, k, C_{\mathcal{Q}, \Phi, p}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}$. \blacklozenge

Proof. We apply Lemma 6.3 with $k+1$ instead of k and with $\phi_i := \varphi_i^{(k+1)*}$. To verify the prerequisites of Lemma 6.3, note that we indeed have $\phi_i \equiv 0$ on $\mathcal{O} \setminus Q_i^{(k+1)*}$ and that Remark 3.16 shows that the family $\Gamma := \left(\varphi_i^{(k+1)*} \right)_{i \in I}$ is indeed an L^p -bounded control system for \mathcal{Q} with

$$C_{\mathcal{Q}, \Gamma, p} \leq C_1 = C_1(\mathcal{Q}, C_{\mathcal{Q}, \Phi, p}, d, p, k).$$

Furthermore, since $\gamma_i \equiv 0$ on $\mathcal{O} \setminus Q_i^{k*}$ and since $\phi_i = \varphi_i^{(k+1)*} \equiv 1$ on Q_i^{k*} (cf. Lemma 2.4), we also get $\phi_i \gamma_i = \gamma_i$ for all $i \in I_0$, as required in Lemma 6.3. The claim now easily follows from that lemma. \square

Above, it was claimed that—using a suitable choice of $z = (z_i)_{i \in I}$ —one can achieve suitable values of $\|f_{z, c, \varepsilon}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)}$, with $f_{z, c, \varepsilon}$ as in equation (6.6). The goal of the next few lemmata is to show how this can be done. The main point is that

$$\left\| \mathcal{F}^{-1} \left[\sum_{i=1}^n M_{z_i} f_i \right] \right\|_{L^p} = \left\| \sum_{i=1}^n L_{-z_i} [\mathcal{F}^{-1} f_i] \right\|_{L^p} \xrightarrow{\min_{i \neq j} |z_i - z_j| \rightarrow \infty} \left\| (\|\mathcal{F}^{-1} f_i\|_{L^p})_{i \in \underline{n}} \right\|_{\ell^p}.$$

Validity of the stated limit follows (at least if $\mathcal{F}^{-1} f_i \in L^p \cap C_0$ for all $i \in \underline{n}$) from the following lemma:

Lemma 6.5. *Let $n \in \mathbb{N}$, $p \in (0, \infty]$ and $f_1, \dots, f_n \in L^p(\mathbb{R}^d)$. For $p = \infty$, assume additionally that*

$$f_i \in \overline{\{f \in L^\infty(\mathbb{R}^d) \mid \text{supp } f \text{ compact}\}} \quad \text{for all } i \in \underline{n},$$

where the closure is taken in $L^\infty(\mathbb{R}^d)$.

Then we have

$$\left\| \sum_{i=1}^n L_{x_i} f_i \right\|_{L^p} \xrightarrow{\min_{i \neq j} |x_i - x_j| \rightarrow \infty} \left\| (\|f_i\|_{L^p})_{i \in \underline{n}} \right\|_{\ell^p}.$$

In particular, there exists $R = R(p, (f_i)_{i \in \underline{n}})$ with

$$\frac{1}{2} \cdot \left\| (\|f_i\|_{L^p})_{i \in \underline{n}} \right\|_{\ell^p} \leq \left\| \sum_{i=1}^n L_{x_i} f_i \right\|_{L^p} \leq 2 \cdot \left\| (\|f_i\|_{L^p})_{i \in \underline{n}} \right\|_{\ell^p}$$

for all $x_1, \dots, x_n \in \mathbb{R}^d$ with $|x_i - x_j| \geq R$ for all $i, j \in \underline{n}$ with $i \neq j$. \blacktriangleleft

Remark. The additional assumption regarding the f_i is certainly satisfied for $f_1, \dots, f_n \in C_0(\mathbb{R}^d)$, where $C_0(\mathbb{R}^d)$ is the space of continuous functions vanishing at infinity, i.e. with $\lim_{|x| \rightarrow \infty} f(x) = 0$. \blacklozenge

Proof. A complete proof of this statement can be found in [24, Lemma 4.1]. Here, we only sketch the idea.

If f_1, \dots, f_n are compactly supported, say $\text{supp } f_i \subset B_R(0)$ for all $i \in \underline{n}$ and if $|x_i - x_j| > 2R$ for all $i \neq j$, then the supports of $(L_{x_i} f_i)_{i \in \underline{n}}$ are pairwise disjoint. Hence, for $p < \infty$, we have

$$\left| \sum_{i=1}^n L_{x_i} f_i(x) \right|^p = \sum_{i=1}^n |L_{x_i} f_i(x)|^p,$$

since for each $x \in \mathbb{R}^d$, at most one summand is nonzero. This easily implies the claim (with equality for $\min_{i \neq j} |x_i - x_j| > 2R$) for compactly supported f_1, \dots, f_n and $p < \infty$. The general case follows by approximating f_1, \dots, f_n by compactly supported $g_1, \dots, g_n \in L^p(\mathbb{R}^d)$. Finally, the case $p = \infty$ can be handled using similar arguments. \square

The following version of the preceding lemma is slightly more adapted to the setting that we will consider.

Corollary 6.6. *Let $p \in (0, \infty]$ and let $M \neq \emptyset$ be a finite index set such that for each $i \in M$ a Schwartz-function $f_i \in \mathcal{S}(\mathbb{R}^d)$ is given.*

For $S \subset M$ and any family $z = (z_i)_{i \in M}$ in \mathbb{R}^d , define

$$f_S^{(z)} := \sum_{i \in S} M_{z_i} f_i \in \mathcal{S}(\mathbb{R}^d).$$

Then there is a constant $R = R((f_i)_{i \in M}, M, p) > 0$ such that for every family $(z_i)_{i \in M}$ in \mathbb{R}^d which satisfies $|z_i - z_j| \geq R$ for all $i, j \in M$ with $i \neq j$, the estimate

$$\frac{1}{2} \cdot \|(\|\mathcal{F}^{-1}f_i\|_{L^p})_{i \in S}\|_{\ell^p} \leq \|\mathcal{F}^{-1}f_S^{(z)}\|_{L^p} \leq 2 \cdot \|(\|\mathcal{F}^{-1}f_i\|_{L^p})_{i \in S}\|_{\ell^p} \quad (6.11)$$

is true for every $S \subset M$. Furthermore, such a family $(z_i)_{i \in M}$ always exists. \blacktriangleleft

Proof. We first show the claim for the case $S = M$. In case of $f_i \equiv 0$ for all $i \in M$, choose $R = 1$ and note that equation (6.11) holds trivially. Otherwise, there is some $i \in M$ with $f_i \not\equiv 0$. In particular, $M \neq \emptyset$, so that we can assume $M = \{i_1, \dots, i_n\}$ with $n := |M|$ and (necessarily) pairwise distinct $i_1, \dots, i_n \in M$.

Because of $\mathcal{F}^{-1}f_i \in \mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$, the number

$$\varepsilon := \frac{1}{2} \cdot \|(\|\mathcal{F}^{-1}f_i\|_{L^p})_{i \in M}\|_{\ell^p} > 0$$

is finite.

For $\ell \in \underline{n}$, let $g_\ell := \mathcal{F}^{-1}f_{i_\ell} \in \mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ and note that we even have $g_\ell \in \mathcal{S}(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$, so that Lemma 6.5 yields some $R > 0$ such that

$$\begin{aligned} \left| \left\| \mathcal{F}^{-1} \left[\sum_{\ell=1}^n M_{-y_\ell} f_{i_\ell} \right] \right\|_{L^p} - 2\varepsilon \right| &= \left| \left\| \sum_{\ell=1}^n L_{y_\ell} (\mathcal{F}^{-1}f_{i_\ell}) \right\|_{L^p} - \|(\|\mathcal{F}^{-1}f_i\|_{L^p})_{i \in M}\|_{\ell^p} \right| \\ &= \left| \left\| \sum_{\ell=1}^n L_{y_\ell} g_\ell \right\|_{L^p} - \|(\|g_\ell\|_{L^p})_{\ell \in \underline{n}}\|_{\ell^p} \right| < \varepsilon \end{aligned}$$

holds for every family $(y_\ell)_{\ell \in \underline{n}}$ which satisfies $|y_\ell - y_k| \geq R$ for all $\ell, k \in \underline{n}$ with $\ell \neq k$. Here, we used the identity

$$\mathcal{F}^{-1}[M_{-z}f] = L_z[\mathcal{F}^{-1}f].$$

For $y_\ell := -z_{i_\ell}$, this implies

$$\|\mathcal{F}^{-1}f_M^{(z)}\|_{L^p} = \left\| \mathcal{F}^{-1} \left[\sum_{\ell=1}^n M_{-y_\ell} f_{i_\ell} \right] \right\|_{L^p} \in (2\varepsilon - \varepsilon, 2\varepsilon + \varepsilon)$$

as soon as we have $|y_k - y_\ell| = |z_{i_k} - z_{i_\ell}| \geq R$ for all $\ell \neq k$. Using the definition of ε , this easily implies the claim for $S = M$.

Now the above (applied to $(f_i)_{i \in S}$ for $S \subset M$) yields some $R_S > 0$ for each $S \subset M$ so that equation (6.11) is satisfied for every family $(z_i)_{i \in S}$ satisfying $|z_i - z_j| \geq R_S$ for all $i, j \in S$ with $i \neq j$.

Since M is finite, the same is true of the power set $\mathcal{P}(M)$, so that $R := \max_{S \subset M} R_S$ is finite. It is now easy to see that the claim holds for this choice of R .

For existence of a family $(z_i)_{i \in M}$ as in the statement of the corollary, simply note that we can take $z_{i_\ell} := \ell \cdot R \cdot (1, 0, \dots, 0) \in \mathbb{R}^d$, since this implies $|z_{i_\ell} - z_{i_k}| = |\ell - k| \cdot R \geq R$ for $\ell, k \in \underline{n}$ with $\ell \neq k$. \square

A convenient tool for choosing suitable signs $\varepsilon = (\varepsilon_i)_{i \in I}$ for the function $f_{z, c, \varepsilon}$ (cf. eq. (6.6)) is given by **Khintchine's inequality** which we state next. A proof can be found e.g. in [25, Proposition 4.5].

Theorem 6.7. Let $p \in (0, \infty)$. Then there is a constant $C_p > 0$ such that

$$C_p^{-1} \cdot \left(\sum_{n=1}^N |a_n|^2 \right)^{p/2} \leq \mathbb{E}_\omega \left| \sum_{n=1}^N \omega_n a_n \right|^p \leq C_p \cdot \left(\sum_{n=1}^N |a_n|^2 \right)^{p/2}$$

holds for all $N \in \mathbb{N}$ and all $a_1, \dots, a_N \in \mathbb{C}$.

Here, the expectation \mathbb{E}_ω is taken with respect to the random variable $\omega \in \{\pm 1\}^N$ which is assumed to be uniformly distributed in that set, i.e. $\mathbb{E}_\omega[f(\omega)] = \frac{1}{2^N} \sum_{\omega \in \{\pm 1\}^N} f(\omega)$. \blacktriangleleft

Using all of these ingredients, we can now prove the following technical lemma which will then imply our desired necessary criteria for the coincidence of two decomposition spaces.

Lemma 6.8. *Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open, let $p \in (0, \infty]$ and let $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ be two open L^p -decomposition coverings of \mathcal{O} , with associated L^p -BAPUs $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_j)_{j \in J}$.*

Finally, let $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ be \mathcal{Q} -regular and \mathcal{P} -regular, respectively, with $\ell_0(I) \leq Y$ and $\ell_0(J) \leq Z$.

Assume that

$$\|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} \asymp \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z)} \quad \forall f \in C_c^\infty(\mathcal{O}). \quad (6.12)$$

Then, for arbitrary $i_0 \in I$ and pairwise distinct $j_1, \dots, j_N \in J_{i_0}$, we have

$$\|\mathbf{1}_{\{j_1, \dots, j_N\}}\|_Z \asymp \|\delta_{i_0}\|_Y \cdot N^{1/p}, \quad (6.13)$$

where the implied constant only depends on the implied constants in equation (6.12) and on

$$d, p, \mathcal{Q}, \mathcal{P}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}, C_{\mathcal{Q}, \Phi, p}, C_{\mathcal{P}, \Psi, p}, C_Z,$$

where $C_Z \geq 1$ is a triangle constant for Z . In particular, the implied constant is independent of i_0, j_1, \dots, j_N and of N .

Furthermore, in case of $p \in (0, \infty)$, we also have

$$\|\mathbf{1}_{\{j_1, \dots, j_N\}}\|_Z \asymp \|\delta_{i_0}\|_Y \cdot N^{1/2}, \quad (6.14)$$

where the implied constant depends on the same quantities as above.

Finally, for $p = \infty$, we have

$$\|\mathbf{1}_{\{j_1, \dots, j_N\}}\|_Z \asymp \|\delta_{i_0}\|_Y \cdot N, \quad (6.15)$$

where the implied constant depends on the same quantities as above. \blacktriangleleft

Proof. Since we assume $\ell_0(I) \leq Y$ and $\ell_0(J) \leq Z$, it is not hard to see that we have

$$C_c^\infty(\mathcal{O}) \subset \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y) = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z),$$

which we will use without comment in the remainder of the proof.

Fix a nontrivial, nonnegative function $\gamma \in C_c^\infty(B_1(0)) \setminus \{0\}$ for the remainder of the proof. Furthermore, set $r_0 := N_{\mathcal{P}}^3 = N_{\mathcal{P}}^{2 \cdot 1 + 1}$, so that the disjointization lemma (Lemma 2.14) yields a finite partition $J = \bigsqcup_{r=1}^{r_0} J^{(r)}$ satisfying $P_j^* \cap P_\ell^* = \emptyset$ for all $j, \ell \in J^{(r)}$ with $j \neq \ell$ and arbitrary $r \in \underline{r_0}$.

Since we assume \mathcal{Q}, \mathcal{P} to be open coverings and because of $j_1, \dots, j_N \in J_{i_0}$, there is some $\varepsilon > 0$ and $\xi_1, \dots, \xi_N \in \mathbb{R}^d$ satisfying $B_\varepsilon(\xi_\ell) \subset Q_{i_0} \cap P_{j_\ell}$ for all $\ell \in \underline{N}$. Note that $\varepsilon > 0$ and ξ_1, \dots, ξ_N depend heavily on j_1, \dots, j_N and on i_0 , but that all occurrences of ε in our estimates will cancel in the end.

Define ξ_j for $j \in J_0 := \{j_1, \dots, j_N\}$ by $\xi_{j_\ell} := \xi_\ell$ for $\ell \in \underline{N}$ and set $\gamma_j^{(\varepsilon)} := L_{\xi_j}[\gamma(\varepsilon^{-1} \cdot)]$ for $j \in J_0$. Note that $\gamma_j^{(\varepsilon)} \in C_c^\infty(B_\varepsilon(\xi_j)) \subset P_j \cap Q_{i_0}$ for all $j \in J_0$. Now, for $r \in \underline{r_0}$, set

$$f_{z, \theta}^{(r)} := \sum_{j \in J_0 \cap J^{(r)}} \theta_j \cdot M_{z_j} \gamma_j^{(\varepsilon)} = \sum_{j \in J_0 \cap J^{(r)}} M_{z_j} [\theta_j \gamma_j^{(\varepsilon)}] \quad \text{for } z = (z_j)_{j \in J_0} \in (\mathbb{R}^d)^{J_0} \text{ and } \theta = (\theta_j)_{j \in J_0} \in \{\pm 1\}^{J_0}.$$

Now, Lemma 6.2 (with \mathcal{P} instead of \mathcal{Q} , with $k = 0$ and with $c_j = \mathbf{1}_{J_0 \cap J^{(r)}}$) yields a constant $C_1 = C_1(d, p, \mathcal{P}, C_{\mathcal{P}, \Psi, p}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}) > 0$ satisfying

$$\begin{aligned} \|f_{z, \theta}^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z)} &\leq C_1 \cdot \left\| \left(\mathbf{1}_{J_0 \cap J^{(r)}}(j) \cdot \|\mathcal{F}^{-1} \gamma_j^{(\varepsilon)}\|_{L^p} \right)_{j \in J} \right\|_Z \\ &= C_1 \cdot \|\mathcal{F}^{-1}[\gamma(\varepsilon^{-1} \cdot)]\|_{L^p} \cdot \|\mathbf{1}_{J_0 \cap J^{(r)}}\|_Z \\ (Z \text{ is solid}) &\leq C_1 \cdot \varepsilon^{d(1-\frac{1}{p})} \|\mathcal{F}^{-1} \gamma\|_{L^p} \cdot \|\mathbf{1}_{J_0}\|_Z, \end{aligned}$$

for arbitrary $r \in \underline{r_0}$. Here, we also used

$$\|\mathcal{F}^{-1} \gamma_j^{(\varepsilon)}\|_{L^p} = \|\mathcal{F}^{-1}(L_{\xi_j}[\gamma(\varepsilon^{-1} \cdot)])\|_{L^p} = \|M_{\xi_j}(\mathcal{F}^{-1}[\gamma(\varepsilon^{-1} \cdot)])\|_{L^p} = \|\mathcal{F}^{-1}[\gamma(\varepsilon^{-1} \cdot)]\|_{L^p},$$

by standard properties of the Fourier transform. Similar identities will be used in the remainder of the proof without comment.

Conversely, Corollary 6.4 (again with \mathcal{P} instead of \mathcal{Q} , with $k = 0$ and with $I_0 = J_0 \cap J^{(r)}$, as well as $c_j = 1$ for all $j \in J_0 \cap J^{(r)} = I_0$) yields a constant $C_2 = C_2(d, p, \mathcal{P}, C_{\mathcal{P}, \Psi, p}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}) > 0$ satisfying

$$\begin{aligned} \|f_{z, \theta}^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z)} &\geq C_2^{-1} \cdot \left\| \left(\|\mathcal{F}^{-1} \gamma_j^{(\varepsilon)}\|_{L^p} \right)_{j \in I_0} \right\|_{Z|_{I_0}} \\ &= C_2^{-1} \cdot \|\mathcal{F}^{-1} [\gamma(\varepsilon^{-1} \cdot)]\|_{L^p} \cdot \|\mathbb{1}_{J_0 \cap J^{(r)}}\|_Z \\ &= C_2^{-1} \cdot \varepsilon^{d(1-\frac{1}{p})} \|\mathcal{F}^{-1} \gamma\|_{L^p} \cdot \|\mathbb{1}_{J_0 \cap J^{(r)}}\|_Z \end{aligned}$$

for arbitrary $r \in \underline{r_0}$. Note that Corollary 6.4 requires $P_j^* \cap P_\ell^* = P_j^{(k+1)*} \cap P_\ell^{(k+1)*} \stackrel{!}{=} \emptyset$ for arbitrary $j, \ell \in I_0 = J_0 \cap J^{(r)}$ with $j \neq \ell$. This holds for every $r \in \underline{r_0}$ by choice of the partition $J = \bigsqcup_{r=1}^{r_0} J^{(r)}$.

Now, using the identity $J_0 = \bigsqcup_{r=1}^{r_0} (J_0 \cap J^{(r)})$ and the (quasi)-triangle inequality for Z , we arrive at $\|\mathbb{1}_{J_0}\|_Z \leq C_3 \cdot \sum_{r=1}^{r_0} \|\mathbb{1}_{J_0 \cap J^{(r)}}\|_Z$ for some constant $C_3 = C_3(r_0, C_Z) = C_3(\mathcal{P}, C_Z)$. Hence,

$$\begin{aligned} \varepsilon^{d(1-\frac{1}{p})} \|\mathcal{F}^{-1} \gamma\|_{L^p} \cdot \|\mathbb{1}_{J_0}\|_Z &\leq C_3 \cdot \sum_{r=1}^{r_0} \varepsilon^{d(1-\frac{1}{p})} \|\mathcal{F}^{-1} \gamma\|_{L^p} \cdot \|\mathbb{1}_{J_0 \cap J^{(r)}}\|_Z \\ &\leq C_2 C_3 \cdot \sum_{r=1}^{r_0} \|f_{z^{(r)}, \theta^{(r)}}^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z)} \\ &\leq C_1 C_2 C_3 r_0 \cdot \varepsilon^{d(1-\frac{1}{p})} \|\mathcal{F}^{-1} \gamma\|_{L^p} \cdot \|\mathbb{1}_{J_0}\|_Z, \end{aligned} \quad (6.16)$$

which is our first main estimate. Note that this estimate holds for *all* choices of $z^{(r)} = (z_j^{(r)})_{j \in J_0}$ and of $\theta^{(r)} = (\theta_j^{(r)})_{j \in J_0}$, where for each $r \in \underline{r_0}$ a different choice is possible.

Now, recall that—by construction— $\text{supp } \gamma_j^{(\varepsilon)} \subset Q_{i_0}$ for all $j \in J_0$, so that also $\text{supp } f_{z, \theta}^{(r)} \subset Q_{i_0}$. Hence, we can apply Lemma 6.2 with $k = 0$, $\gamma_{i_0} = f_{z, \theta}^{(r)}$ and $\gamma_i \equiv 0$ for all $i \in I \setminus \{i_0\}$, as well as $c_i = \delta_{i_0}(i)$, $z_i = 0$ and $\varepsilon_i = 1$ for all $i \in I$ to conclude

$$\|f_{z, \theta}^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} \leq C_4 \cdot \|\mathcal{F}^{-1} f_{z, \theta}^{(r)}\|_{L^p} \cdot \|\delta_{i_0}\|_Y$$

for some constant $C_4 = C_4(d, p, \mathcal{Q}, C_{\mathcal{Q}, \Phi, p}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y})$ and arbitrary choice of z, θ . Note that Lemma 6.2 is indeed applicable with the above choices, since we have $\delta_{i_0} \in \ell_0(I) \leq Y$.

Conversely, we can also apply Corollary 6.4 with $I_0 = \{i_0\}$, $\gamma_{i_0} = f_{z, \theta}^{(r)}$ and $c_{i_0} = \varepsilon_{i_0} = 1$ and $z_{i_0} = 0$ to conclude

$$\|f_{z, \theta}^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} \geq C_5^{-1} \cdot \|\mathcal{F}^{-1} f_{z, \theta}^{(r)}\|_{L^p} \cdot \|\delta_{i_0}\|_Y$$

for a further constant $C_5 = C_5(p, d, \mathcal{Q}, C_{\mathcal{Q}, \Phi, p}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y})$ and arbitrary choices of z, θ .

In summary with equation (6.16) and with our assumption $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} \asymp \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z)}$, we have thus shown

$$\begin{aligned} \varepsilon^{d(1-\frac{1}{p})} \|\mathcal{F}^{-1} \gamma\|_{L^p} \cdot \|\mathbb{1}_{J_0}\|_Z &\asymp \sum_{r=1}^{r_0} \|f_{z^{(r)}, \theta^{(r)}}^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z)} \\ &\asymp \sum_{r=1}^{r_0} \|f_{z^{(r)}, \theta^{(r)}}^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} \\ &\asymp \|\delta_{i_0}\|_Y \cdot \sum_{r=1}^{r_0} \|\mathcal{F}^{-1} f_{z^{(r)}, \theta^{(r)}}^{(r)}\|_{L^p}, \end{aligned} \quad (6.17)$$

where the implied constants only depend on $p, d, C_Z, r_0, \mathcal{Q}, \mathcal{P}, C_{\mathcal{Q}, \Phi, p}, C_{\mathcal{P}, \Psi, p}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}$ and on the implied constants in $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} \asymp \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z)}$.

With estimate (6.17) we have found an opportunity for *arbitrage*: The left-hand side is independent of $z^{(r)}, \theta^{(r)}$, while the right-hand side is not, as we will see now: If we choose $\theta_j^{(r)} = 1$ for all $j \in J_0$,

we get using Corollary (6.6) (which is applicable since $\gamma_j^{(\varepsilon)} \in \mathcal{S}(\mathbb{R}^d)$ for all $j \in J_0$) that

$$\begin{aligned} \left\| \mathcal{F}^{-1} f_{z^{(r)}, \theta^{(r)}}^{(r)} \right\|_{L^p} &= \left\| \mathcal{F}^{-1} \left[\sum_{j \in J_0 \cap J^{(r)}} M_{z_j^{(r)}} \gamma_j^{(\varepsilon)} \right] \right\|_{L^p} \\ &\left(\text{for suitable } (z_j^{(r)})_{j \in J_0} \in (\mathbb{R}^d)^{J_0} \right) \asymp \left\| \left(\left\| \mathcal{F}^{-1} \gamma_j^{(\varepsilon)} \right\|_{L^p} \right)_{j \in J_0 \cap J^{(r)}} \right\|_{\ell^p} \\ &= \varepsilon^{d(1-\frac{1}{p})} \left\| \mathcal{F}^{-1} \gamma \right\|_{L^p} \cdot \left| J_0 \cap J^{(r)} \right|^{1/p}. \end{aligned}$$

Now, note that we have $|J_0 \cap J^{(r)}| \leq |J_0| = N$ for all $r \in \underline{r_0}$ and that $J_0 = \bigsqcup_{r=1}^{r_0} (J_0 \cap J^{(r)})$, so that there is some $r \in \underline{r_0}$ with $|J_0 \cap J^{(r)}| \geq |J_0|/r_0 = N/r_0$. Thus, using equation (6.17), we get (for suitable choices of the coefficients $z^{(r)}$ and $\theta^{(r)}$) that

$$\begin{aligned} \varepsilon^{d(1-\frac{1}{p})} \left\| \mathcal{F}^{-1} \gamma \right\|_{L^p} \cdot \left\| \mathbf{1}_{J_0} \right\|_Z &\asymp \left\| \delta_{i_0} \right\|_Y \cdot \sum_{r=1}^{r_0} \left[\varepsilon^{d(1-\frac{1}{p})} \left\| \mathcal{F}^{-1} \gamma \right\|_{L^p} \cdot \left| J_0 \cap J^{(r)} \right|^{1/p} \right] \\ &\asymp \varepsilon^{d(1-\frac{1}{p})} \left\| \mathcal{F}^{-1} \gamma \right\|_{L^p} \cdot \left\| \delta_{i_0} \right\|_Y \cdot N^{1/p}. \end{aligned} \quad (6.18)$$

Now, we can cancel the common factor $\varepsilon^{d(1-\frac{1}{p})} \left\| \mathcal{F}^{-1} \gamma \right\|_{L^p}$ on both sides to obtain equation (6.13). Note in particular that all occurrences of ε canceled.

Next, let us consider the case $p \in (0, \infty)$. In this case, choose $z_j^{(r)} = 0$ for all $j \in J_0$ and let $\omega = (\omega_j)_{j \in J_0}$ be a random variable which is uniformly distributed in $\{\pm 1\}^{J_0}$. The expected value in the following calculation is taken with respect to ω . Elementary properties of the Fourier transform and Khintchine's inequality (Theorem 6.7) yield

$$\begin{aligned} \mathbb{E} \left\| \mathcal{F}^{-1} f_{z^{(r)}, \omega}^{(r)} \right\|_{L^p}^p &= \mathbb{E} \int_{\mathbb{R}^d} \left| \sum_{j \in J_0 \cap J^{(r)}} \omega_j \cdot \left(\mathcal{F}^{-1} \gamma_j^{(\varepsilon)} \right) (x) \right|^p dx \\ &= \mathbb{E} \int_{\mathbb{R}^d} \left| \sum_{j \in J_0 \cap J^{(r)}} \omega_j \cdot e^{2\pi i \langle \xi_j, x \rangle} \cdot \varepsilon^d (\mathcal{F}^{-1} \gamma) (\varepsilon x) \right|^p dx \\ &= \varepsilon^{dp} \cdot \int_{\mathbb{R}^d} \left| (\mathcal{F}^{-1} \gamma) (\varepsilon x) \right|^p \cdot \mathbb{E} \left| \sum_{j \in J_0 \cap J^{(r)}} \omega_j \cdot e^{2\pi i \langle \xi_j, x \rangle} \right|^p dx \\ &\stackrel{(\text{Theorem 6.7})}{\asymp} \varepsilon^{dp} \cdot \int_{\mathbb{R}^d} \left| (\mathcal{F}^{-1} \gamma) (\varepsilon x) \right|^p \cdot \left(\sum_{j \in J_0 \cap J^{(r)}} \left| e^{2\pi i \langle \xi_j, x \rangle} \right|^2 \right)^{p/2} dx \\ &= \varepsilon^{dp} \cdot \left| J_0 \cap J^{(r)} \right|^{p/2} \cdot \int_{\mathbb{R}^d} \left| (\mathcal{F}^{-1} \gamma) (\varepsilon x) \right|^p dx \\ &= \left(\varepsilon^{d(1-\frac{1}{p})} \cdot \left| J_0 \cap J^{(r)} \right|^{1/2} \cdot \left\| \mathcal{F}^{-1} \gamma \right\|_{L^p} \right)^p, \end{aligned} \quad (6.19)$$

where the implied constant only depends on $p \in (0, \infty)$. Note that we could interchange the expectation and the integral in the preceding calculation, since the expectation is just a finite sum. Alternatively, we could have used Fubini's theorem.

In particular, estimate (6.19) yields deterministic realizations $\theta^{(U,r)} = \left(\theta_j^{(U,r)} \right)_{j \in J_0} \in \{\pm 1\}^{J_0}$ and $\theta^{(L,r)} = \left(\theta_j^{(L,r)} \right)_{j \in J_0} \in \{\pm 1\}^{J_0}$ satisfying

$$\left\| \mathcal{F}^{-1} f_{z^{(L,r)}, \theta^{(r)}}^{(r)} \right\|_{L^p} \lesssim \varepsilon^{d(1-\frac{1}{p})} \left\| \mathcal{F}^{-1} \gamma \right\|_{L^p} \cdot \left| J_0 \cap J^{(r)} \right|^{1/2} \lesssim \left\| \mathcal{F}^{-1} f_{z^{(U,r)}, \theta^{(r)}}^{(r)} \right\|_{L^p},$$

where the implied constants again only depend on $p \in (0, \infty)$. Using essentially the same arguments as before equation (6.18), we thus conclude

$$\begin{aligned} \varepsilon^{d(1-\frac{1}{p})} \|\mathcal{F}^{-1}\gamma\|_{L^p} \cdot \|\delta_{i_0}\|_Y \cdot N^{1/2} &\lesssim \|\delta_{i_0}\|_Y \cdot \sum_{r=1}^{r_0} \left\| \mathcal{F}^{-1} f_{z^{(r)}, \theta^{(r)}}^{(r)} \right\|_{L^p} \\ &\asymp \varepsilon^{d(1-\frac{1}{p})} \|\mathcal{F}^{-1}\gamma\|_{L^p} \cdot \|\mathbb{1}_{J_0}\|_Z \\ &\asymp \|\delta_{i_0}\|_Y \cdot \sum_{r=1}^{r_0} \left\| \mathcal{F}^{-1} f_{z^{(r)}, \theta^{(r)}}^{(r)} \right\|_{L^p} \\ &\lesssim \varepsilon^{d(1-\frac{1}{p})} \|\mathcal{F}^{-1}\gamma\|_{L^p} \cdot \|\delta_{i_0}\|_Y \cdot N^{1/2}, \end{aligned}$$

which establishes equation (6.14).

Finally, in case of $p = \infty$, we choose $z_j^{(r)} = 0$ and $\theta_j^{(r)} = 1$ for all $j \in J_0$ to arrive at

$$\begin{aligned} \left\| \mathcal{F}^{-1} f_{z^{(r)}, \theta^{(r)}}^{(r)} \right\|_{L^p} &= \left\| \sum_{j \in J_0 \cap J^{(r)}} \mathcal{F}^{-1} \gamma_j^{(\varepsilon)} \right\|_{L^\infty} \\ &\geq \left| \sum_{j \in J_0 \cap J^{(r)}} \left(\mathcal{F}^{-1} \gamma_j^{(\varepsilon)} \right) (0) \right| \\ &= \left(\text{since } \gamma_j^{(\varepsilon)} \geq 0 \right) = \sum_{j \in J_0 \cap J^{(r)}} \left\| \gamma_j^{(\varepsilon)} \right\|_{L^1} \\ &\stackrel{\text{(Riemann-Lebesgue)}}{\geq} \sum_{j \in J_0 \cap J^{(r)}} \left\| \mathcal{F}^{-1} \gamma_j^{(\varepsilon)} \right\|_{L^\infty}. \end{aligned}$$

The reverse estimate $\left\| \mathcal{F}^{-1} f_{z^{(r)}, \theta^{(r)}}^{(r)} \right\|_{L^p} \leq \sum_{j \in J_0 \cap J^{(r)}} \left\| \mathcal{F}^{-1} \gamma_j^{(\varepsilon)} \right\|_{L^\infty}$ is a direct consequence of the triangle inequality. Hence, recalling $p = \infty$, we get

$$\left\| \mathcal{F}^{-1} f_{z^{(r)}, \theta^{(r)}}^{(r)} \right\|_{L^p} = \sum_{j \in J_0 \cap J^{(r)}} \left\| \mathcal{F}^{-1} \gamma_j^{(\varepsilon)} \right\|_{L^\infty} = \varepsilon^{d(1-\frac{1}{p})} \cdot \|\mathcal{F}^{-1}\gamma\|_{L^p} \cdot |J_0 \cap J^{(r)}|.$$

Exactly as in equation (6.18), this yields

$$\varepsilon^{d(1-\frac{1}{p})} \|\mathcal{F}^{-1}\gamma\|_{L^p} \cdot \|\mathbb{1}_{J_0}\|_Z \asymp \|\delta_{i_0}\|_Y \cdot \sum_{r=1}^{r_0} \left\| \mathcal{F}^{-1} f_{z^{(r)}, \theta^{(r)}}^{(r)} \right\|_{L^p} \asymp \varepsilon^{d(1-\frac{1}{p})} \|\mathcal{F}^{-1}\gamma\|_{L^p} \cdot \|\delta_{i_0}\|_Y \cdot N,$$

which establishes equation (6.15). \square

Now, we can finally prove the announced necessary criteria for the equality of two decomposition spaces:

Theorem 6.9. Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open, let $p_1, p_2 \in (0, \infty]$ and assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an *open* L^{p_1} -decomposition covering of \mathcal{O} and that $\mathcal{P} = (P_j)_{j \in J}$ is an *open* L^{p_2} -decomposition covering of \mathcal{O} . Let $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ be \mathcal{Q} -regular and \mathcal{P} -regular, respectively and assume $\ell_0(I) \leq Y$ and $\ell_0(J) \leq Z$.

Finally, assume that

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$$

holds, with equivalent (quasi)-norms. Then the following hold:

- (1) We have $p_1 = p_2$.
- (2) There is a constant $C_1 > 0$ with

$$C_1^{-1} \cdot \|\delta_i\|_Y \leq \|\delta_j\|_Z \leq C_1 \cdot \|\delta_i\|_Y \text{ for all } i \in I \text{ and } j \in J \text{ with } Q_i \cap P_j \neq \emptyset.$$

- (3) In case of $p_1 \neq 2$, \mathcal{Q} and \mathcal{P} are weakly equivalent (cf. Definition 2.10).
- (4) In case of $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$ for a \mathcal{Q} -moderate weight $w = (w_i)_{i \in I}$ and a \mathcal{P} -moderate weight $v = (v_j)_{j \in J}$ and certain $q_1, q_2 \in (0, \infty]$, the following hold:
 - (a) We have $q_1 = q_2$.

(b) There is a constant $C_1 > 0$ with

$$C_1^{-1} \cdot w_i \leq v_j \leq C_1 \cdot w_i \text{ for all } i \in I \text{ and } j \in J \text{ with } Q_i \cap P_j \neq \emptyset.$$

(c) If $(p_1, q_1) \neq (2, 2)$, the coverings \mathcal{Q}, \mathcal{P} are weakly equivalent. \blacktriangleleft

Remark. 1. Note that the theorem only yields *weak* equivalence of \mathcal{Q}, \mathcal{P} . Nevertheless, if all sets of the coverings \mathcal{Q}, \mathcal{P} are connected (which is usually the case), then these sets are even path-connected, since both notions of connectedness are equivalent for open subsets of \mathbb{R}^d and since we assume \mathcal{Q}, \mathcal{P} to be open coverings. But then, Corollary 2.13 shows that \mathcal{Q}, \mathcal{P} are equivalent, not only weakly equivalent.

2. It is not necessary to assume that the (quasi)-norms on both spaces are equivalent. Indeed, using the completeness of decomposition spaces, a form of the closed graph theorem (cf. [16, Theorem 2.15]) and the continuous embedding $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}'(\mathcal{O})$ (cf. Theorem 3.21), it is not hard to see that an equality of the two decomposition spaces as sets already implies that the (quasi)-norms are equivalent.

3. Instead of assuming $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$, the proof shows that it suffices to assume $\|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)} \asymp \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)}$ for all $f \in C_c^\infty(\mathcal{O})$.

4. The requirement that \mathcal{Q}, \mathcal{P} are *open* coverings is made mostly for convenience. It is not hard to see that we could always replace \mathcal{Q} and \mathcal{P} by the open coverings \mathcal{Q}° and \mathcal{P}° and get the same decomposition spaces. The main reason for this is that for any subordinate partition of unity $(\varphi_i)_{i \in I}$, we already have $\varphi_i \equiv 0$ outside of Q_i° . Note though that $Q_i \cap P_j \neq \emptyset$ is not necessarily equivalent to $Q_i^\circ \cap P_j^\circ \neq \emptyset$. To avoid distinctions of this type, we directly assume \mathcal{Q} and \mathcal{P} to be open.

5. In contrast to our other theorems, the conditions in the theorem above are all stated *qualitatively*, e.g. no (explicit) bounds on the quantities $N(\mathcal{Q}, \mathcal{P})$ and $N(\mathcal{P}, \mathcal{Q})$ are provided. As the proof shows, it is still true that we have $N(\mathcal{Q}, \mathcal{P}) \leq C$ and $N(\mathcal{P}, \mathcal{Q}) \leq C$, where $C > 0$ depends on the bounds of the norm equivalence $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)} \asymp \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)}$ and on the usual quantities like $p_1, q_1, \mathcal{Q}, \mathcal{P}$, etc. In this case, we prefer the compact form of the theorem, since we will not need to know the precise dependency of the constants in any relevant application. \blacklozenge

Proof of Theorem 6.9. A straightforward application of Lemma 6.1 (with $K = \mathcal{O}$) shows that we have $p_1 \leq p_2$ and $p_2 \leq p_1$ and thus $p_1 = p_2$. Given this identity, the same lemma also yields existence of $C_1 > 0$ as required in (2) and (4b). This uses that (by assumption) $\delta_i \in Y$ and $\delta_j \in Z$ for arbitrary $i \in I$ and $j \in J$ and that \mathcal{Q}, \mathcal{P} are open coverings of \mathcal{O} , so that $Q_i \cap P_j \neq \emptyset$ is equivalent to $K^\circ \cap Q_i^\circ \cap P_j^\circ \neq \emptyset$. Furthermore, we used $\|\delta_i\|_{\ell_w^{q_1}} = w_i$ and $\|\delta_j\|_{\ell_v^{q_2}} = v_j$ for all $i \in I$ and $j \in J$.

Let us first show that \mathcal{Q}, \mathcal{P} are weakly equivalent if $p_1 \neq 2$. Note that (because of $p_1 = p_2$), our assumptions are symmetric in \mathcal{Q}, \mathcal{P} , so that it suffices to show that \mathcal{Q} is weakly subordinate to \mathcal{P} , i.e. that $\sup_{i \in I} |J_i| < \infty$. To this end, first assume $p_1 \in (0, \infty)$ and let $i_0 \in I$ be arbitrary. For pairwise distinct $j_1, \dots, j_N \in J_{i_0}$, Lemma 6.8 (precisely, equations (6.13) and (6.14)) yields

$$\|\delta_{i_0}\|_Y \cdot N^{1/p_1} \asymp \|\mathbb{1}_{\{j_1, \dots, j_N\}}\|_Z \asymp \|\delta_{i_0}\|_Y \cdot N^{1/2}$$

and hence $N^{1/p_1} \asymp N^{1/2}$, where the implied constants are independent of $i_0 \in I$ and of N . Since we assume $p_1 \neq 2$, it is easy to see that this can only hold if $N \leq C$ for some absolute constant $C > 0$. This yields $\sup_{i \in I} |J_i| < \infty$ as desired.

Finally, in case of $p_1 = \infty$, we again use Lemma 6.8 (this time equations (6.13) and (6.15)) to deduce $1 = N^{1/p_1} \asymp N$, which again yields $\sup_{i \in I} |J_i| < \infty$, as desired.

We have now completed the proof of the first three claims of the theorem, so that we can assume $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$ for the remainder of the proof.

For brevity, set $p := p_1 = p_2$. Let $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_j)_{j \in J}$ be L^p -BAPUs for \mathcal{Q} and \mathcal{P} , respectively. Furthermore, fix a nontrivial, nonnegative function $\gamma \in C_c^\infty(B_1(0))$ for the rest of the proof. For $\varepsilon > 0$, define $\gamma^{(\varepsilon)} := \gamma(\varepsilon^{-1} \cdot) \in C_c^\infty(B_\varepsilon(0))$. Finally, fix $C \geq 1$ with

$$C^{-1} \cdot \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, \ell_w^{q_1})} \leq \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, \ell_v^{q_2})} \leq C \cdot \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, \ell_w^{q_1})} \quad \forall f \in C_c^\infty(\mathcal{O}).$$

Let us first show $q_1 = q_2$. By symmetry, it suffices to show $q_1 \leq q_2$. To this end, let $N \in \mathbb{N}$ be arbitrary and assume that there are sequences $i_1, \dots, i_N \in I$ and $j_1, \dots, j_N \in J$ with $Q_{i_\ell} \cap P_{j_\ell} \neq \emptyset$ for

all $\ell \in \underline{N}$ and with $Q_{i_\ell}^* \cap Q_{i_k}^* = \emptyset = P_{j_\ell}^* \cap P_{j_k}^*$ for all $\ell, k \in \underline{N}$ with $\ell \neq k$. We will see below that such a sequence indeed exists. Since the family $(Q_{i_\ell} \cap P_{j_\ell})_{\ell \in \underline{N}}$ is a finite family of nonempty open(!) subsets of \mathcal{O} , there is some $\varepsilon > 0$ and a sequence $\xi_1, \dots, \xi_N \in \mathcal{O}$ with $B_\varepsilon(\xi_\ell) \subset Q_{i_\ell} \cap P_{j_\ell}$ for all $\ell \in \underline{N}$. Note that $\varepsilon > 0$ may depend heavily on $N \in \mathbb{N}$ and on i_1, \dots, i_N and j_1, \dots, j_N , but—as we will see—all dependencies on ε that are relevant for us will cancel in the end.

As an auxiliary result, note that $\ell \neq k$ implies $i_\ell \neq i_k$; indeed, we have $Q_{i_\ell}^* \cap Q_{i_k}^* = \emptyset$ since $\ell \neq k$. But $i_\ell = i_k$ would imply $Q_{i_\ell}^* \cap Q_{i_k}^* = Q_{i_\ell}^* \supset Q_{i_\ell} \neq \emptyset$, since $Q_{i_\ell} \cap P_{j_\ell} \neq \emptyset$. The same argument also shows that $\ell \neq k$ implies $j_\ell \neq j_k$.

Now, let $\gamma_\ell^{(\varepsilon)} := L_{\xi_\ell} \gamma_\ell \in C_c^\infty(B_\varepsilon(\xi_\ell))$ for $\ell \in \underline{N}$ and define

$$f_N := \sum_{\ell=1}^N w_{i_\ell}^{-1} \cdot \gamma_\ell^{(\varepsilon)} \in C_c^\infty(\mathcal{O}) \subset \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, \ell_w^{q_1}) \cap \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, \ell_v^{q_2}).$$

By assumption, we have

$$\|f_N\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, \ell_v^{q_2})} \leq C \cdot \|f_N\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, \ell_w^{q_1})}. \quad (6.20)$$

To exploit this estimate, we will now obtain an upper bound on $\|f_N\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, \ell_w^{q_1})}$ and a lower bound on $\|f_N\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, \ell_v^{q_2})}$. Since each $\gamma_\ell^{(\varepsilon)}$ vanishes outside of $B_\varepsilon(\xi_\ell) \subset Q_{i_\ell}$ (and since we have $i_\ell \neq i_m$ for $\ell \neq m$), Lemma 6.2 yields a constant $L_1 > 0$ (which does *not* depend on N) with

$$\begin{aligned} \|f_N\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, \ell_w^{q_1})} &\leq L_1 \cdot \left\| \left(w_{i_\ell} \cdot \left\| \mathcal{F}^{-1} \left[w_{i_\ell}^{-1} \cdot \gamma_\ell^{(\varepsilon)} \right] \right\|_{L^p} \right)_{\ell \in \underline{N}} \right\|_{\ell^{q_1}} \\ &= L_1 \cdot \left\| \left(\varepsilon^{d(1-\frac{1}{p})} \cdot \left\| \mathcal{F}^{-1} \gamma \right\|_{L^p} \right)_{\ell \in \underline{N}} \right\|_{\ell^{q_1}} \\ &= L_1 \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \left\| \mathcal{F}^{-1} \gamma \right\|_{L^p} \cdot N^{1/q_1}. \end{aligned} \quad (6.21)$$

Likewise, Corollary 6.4 (with \mathcal{P} instead of \mathcal{Q} and with $I_0 = \{j_1, \dots, j_N\}$ and $k = 0$) yields a constant $L_2 > 0$ (independent of $N \in \mathbb{N}$) with

$$\begin{aligned} \|f_N\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, \ell_v^{q_2})} &\geq L_2^{-1} \cdot \left\| \left(v_{j_\ell} \cdot \left\| \mathcal{F}^{-1} \left(w_{i_\ell}^{-1} \cdot \gamma_\ell^{(\varepsilon)} \right) \right\|_{L^p} \right)_{\ell \in \underline{N}} \right\|_{\ell^{q_2}} \\ (v_{j_\ell} \geq C_1^{-1} \cdot w_{i_\ell} \text{ since } Q_{i_\ell} \cap P_{j_\ell} \neq \emptyset) &\geq (C_1 L_2)^{-1} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \left\| \mathcal{F}^{-1} \gamma \right\|_{L^p} \cdot N^{1/q_2}. \end{aligned} \quad (6.22)$$

Here, the prerequisites of Corollary 6.4 are indeed satisfied, since we have $\text{supp } \gamma_\ell^{(\varepsilon)} \subset P_{j_\ell} = P_{j_\ell}^{k*}$ for all $\ell \in \underline{N}$ and because of $P_{j_\ell}^{(k+1)*} \cap P_{j_n}^{(k+1)*} = P_{j_\ell}^* \cap P_{j_n}^* = \emptyset$ as soon as $j_\ell \neq j_n$ (which implies $\ell \neq n$). Finally, we also used $j_\ell \neq j_n$ for $\ell \neq n$ —which was shown above—since this implies that the map $I_0 = \{j_1, \dots, j_N\} \rightarrow C_c^\infty(\mathcal{O}), j_\ell \mapsto w_{i_\ell}^{-1} \cdot \gamma_\ell^{(\varepsilon)}$ is well-defined.

Finally, a combination of inequalities (6.20), (6.21) and (6.22) yields—after canceling the common factor $\varepsilon^{d(1-\frac{1}{p})} \cdot \left\| \mathcal{F}^{-1} \gamma \right\|_{L^p} > 0$ —the estimate

$$N^{1/q_2} \leq C C_1 L_1 L_2 \cdot N^{1/q_1}.$$

Since the constant $C C_1 L_1 L_2$ is independent of N and since this estimate holds for all $N \in \mathbb{N}$, we conclude $q_2^{-1} \leq q_1^{-1}$ and thus $q_1 \leq q_2$ as desired.

To complete the proof of $q_1 = q_2$, it remains to construct sequences of indices i_1, \dots, i_N and j_1, \dots, j_N with $Q_{i_\ell} \cap P_{j_\ell} \neq \emptyset$ for all $\ell \in \underline{N}$ and so that $(Q_{i_\ell}^*)_{\ell \in \underline{N}}$ and $(P_{j_\ell}^*)_{\ell \in \underline{N}}$ are two sequences of pairwise disjoint sets. This is done by induction on $N \in \mathbb{N}$: If i_1, \dots, i_{N-1} and j_1, \dots, j_{N-1} are already constructed, note that

$$M := \bigcup_{\ell=1}^{N-1} \overline{Q_{i_\ell}^{3*} \cup P_{j_\ell}^{3*}}$$

is a compact subset of \mathcal{O} by Lemma 2.4. Since $\mathcal{O} \subset \mathbb{R}^d$ is a nonempty open set, we conclude¹⁰ $M \subsetneq \mathcal{O}$. Thus, there is some $\xi \in \mathcal{O} \setminus M$. Since \mathcal{Q}, \mathcal{P} both cover \mathcal{O} , there are $i_N \in I$ and $j_N \in J$

¹⁰Otherwise, $M = \mathcal{O}$ would be compact and open, so that connectedness of \mathbb{R}^d yields $M = \mathcal{O} \in \{\emptyset, \mathbb{R}^d\}$, which is impossible, since \mathcal{O} is nonempty and M is compact.

with $\xi \in Q_{i_N} \cap P_{j_N} \neq \emptyset$. It remains to verify that $(Q_{i_\ell}^*)_{\ell \in \underline{N}}$ and $(P_{j_\ell}^*)_{\ell \in \underline{N}}$ are both pairwise disjoint. By symmetry, it suffices to consider the first sequence. Also, since $(Q_{i_\ell}^*)_{\ell \in \underline{N-1}}$ is pairwise disjoint, we only need to show $Q_{i_N}^* \cap Q_{i_\ell}^* = \emptyset$ for all $\ell \in \underline{N-1}$. If this was false, we would have $Q_i \cap Q_{i_\ell}^* \neq \emptyset$ for some $i \in i_N^*$. Hence, $i \in i_\ell^{2*}$, which implies $i_N \in i_\ell^{3*}$ and thus $\xi \in Q_{i_N} \subset Q_{i_\ell}^{3*} \subset M$, in contradiction to $\xi \in \mathcal{O} \setminus M$. All in all, we have thus shown $q_1 = q_2$. For brevity, let us set $q := q_1$ for the remainder of the proof.

It remains to show that \mathcal{Q} and \mathcal{P} are weakly equivalent if $(p_1, q_1) \neq (2, 2)$. In case of $p_1 \neq 2$, this follows from our general considerations above. Hence, we can assume $p_1 = 2$ and thus $q_1 = q_2 \neq 2$, since $(p_1, q_1) \neq (2, 2)$. As above, our assumptions are symmetric in \mathcal{Q}, \mathcal{P} , so that it suffices to show that \mathcal{Q} is weakly subordinate to \mathcal{P} , i.e. that $\sup_{i \in I} |J_i| < \infty$. To this end, let $i_0 \in I$ be arbitrary and let $j_1, \dots, j_N \in J_{i_0}$ be pairwise distinct.

Note that we have $p_1 = 2 < \infty$. Thus, Lemma 6.8 (precisely, equation (6.14)) shows that

$$\begin{aligned} w_{i_0} \cdot N^{1/2} &= \|\delta_{i_0}\|_Y \cdot N^{1/2} \asymp \|\mathbb{1}_{\{j_1, \dots, j_N\}}\|_Z \\ &= \left\| (v_j \cdot \mathbb{1}_{\{j_1, \dots, j_N\}}(j))_{j \in J} \right\|_{\ell_{q_2}} \\ (v_j &\asymp w_{i_0} \text{ for } j \in \{j_1, \dots, j_N\} \subset J_{i_0}) \asymp w_{i_0} \cdot \|\mathbb{1}_{\{j_1, \dots, j_N\}}\|_{\ell_{q_2}} \\ &(\text{since } q_2 = q_1) = w_{i_0} \cdot N^{1/q_1}, \end{aligned}$$

where the implied constants are independent of $i_0 \in I$ and of N . But since we assume $q_1 \neq 2$, this is only possible if $N \leq C$, for some absolute constant $C > 0$. This shows $\sup_{i \in I} |J_i| < \infty$, as desired. \square

We close this subsection by showing that the necessary criteria from Theorem 6.9 for the equality of two decomposition spaces—with weighted Lebesgue sequence spaces as global components—are sufficient for the coincidence of the two decomposition spaces. At least for $p \in [1, \infty]$, this is true as stated. In the Quasi-Banach regime $p \in (0, 1)$, we will need to impose certain additional restrictions.

As our starting point, note that Theorem 6.9 only shows that \mathcal{Q}, \mathcal{P} are weakly equivalent as long as $(p_1, q_1) \neq (2, 2)$. This is natural, since all (Fourier side) decomposition spaces $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, \ell_w^q)$ with $p = q = 2$ are just certain weighted L^2 -spaces:

Lemma 6.10. *Let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open and assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an L^2 -decomposition covering of \mathcal{O} . Finally, let $w = (w_i)_{i \in I}$ be \mathcal{Q} -moderate. Then there is a measurable weight*

$$w_0 : \mathcal{O} \rightarrow (0, \infty) \quad \text{with} \quad w_0(\xi) \asymp w_i \text{ for all } \xi \in Q_i \text{ and arbitrary } i \in I,$$

where the implied constant is independent of i, ξ .

For each such weight w_0 , both w_0 and w_0^{-1} are locally bounded on \mathcal{O} . Furthermore, we have

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^2, \ell_w^2) = L_{w_0}^2(\mathcal{O}), \quad (6.23)$$

with equivalent norms.

In particular, if $\mathcal{P} = (P_j)_{j \in J}$ is another L^2 -decomposition covering of \mathcal{O} and if $v = (v_j)_{j \in J}$ is \mathcal{P} -moderate with $w_i \asymp v_j$ in case of $Q_i \cap P_j \neq \emptyset$, then

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^2, \ell_w^2) = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^2, \ell_v^2)$$

with equivalent norms. \blacktriangleleft

Proof. Let $\Phi = (\varphi_i)_{i \in I}$ be an L^2 -BAPU for \mathcal{O} . By Lemma 2.4, Φ is a locally finite partition of unity on \mathcal{O} . Since $\mathcal{O} \subset \mathbb{R}^d$ is second countable, this easily shows $\varphi_i \equiv 0$ for all $i \in I \setminus I_0$ for some countable set $I_0 \subset I$.

Furthermore, since Φ is a smooth, locally finite partition of unity on \mathcal{O} , we see that

$$w_0 : \mathcal{O} \rightarrow \mathbb{R}, \xi \mapsto \sum_{i \in I} w_i \varphi_i(\xi) \quad (6.24)$$

is smooth. For $\ell \in I$ and $\xi \in Q_\ell$, we have $\varphi_i(\xi) = 0$ unless $i \in \ell^*$. But in case of $i \in \ell^*$, we have $C_{w, \mathcal{Q}}^{-1} \cdot w_\ell \leq w_i \leq C_{w, \mathcal{Q}} \cdot w_\ell$. Thus,

$$0 < C_{w, \mathcal{Q}}^{-1} \cdot w_\ell \leq C_{w, \mathcal{Q}}^{-1} \cdot w_\ell \sum_{i \in I} \varphi_i(\xi) \leq w_0(\xi) \leq C_{w, \mathcal{Q}} \cdot w_\ell \sum_{i \in I} \varphi_i(\xi) = C_{w, \mathcal{Q}} \cdot w_\ell.$$

We have thus shown $w_0(\xi) \asymp w_\ell$ for $\xi \in Q_\ell$ and $\ell \in I$, where the implied constant only depends on $C_{w, \mathcal{Q}}$. In particular, $w_0 : \mathcal{O} \rightarrow (0, \infty)$, as desired.

Now, let $w_0 : \mathcal{O} \rightarrow (0, \infty)$ be an arbitrary (measurable) weight with

$$C^{-1} \cdot w_i \leq w_0(\xi) \leq C \cdot w_i \quad \forall i \in I \forall \xi \in Q_i$$

for some constant $C \geq 1$. For arbitrary $\xi \in \mathcal{O}$, Lemma 2.4 shows that there is some $i \in I$ with $\xi \in Q_i^\circ$. But on this set, we have $w_0(\eta) \leq C \cdot w_i$ and $[w_0(\eta)]^{-1} \leq C \cdot w_i^{-1}$, so that w_0 and w_0^{-1} are bounded on the neighborhood Q_i° of ξ . Hence, w_0 and w_0^{-1} are locally bounded on \mathcal{O} .

For the proof of “ \hookleftarrow ” in equation (6.23), let $f \in L_{w_0}^2(\mathcal{O})$ be arbitrary. Since w_0^{-1} is locally bounded, w_0 is locally bounded from below, so that we get

$$L_{w_0}^2(\mathcal{O}) \hookrightarrow L_{\text{loc}}^2(\mathcal{O}) \hookrightarrow L_{\text{loc}}^1(\mathcal{O}) \hookrightarrow \mathcal{D}'(\mathcal{O}).$$

Now, continuity of φ_i implies

$$|\varphi_i(\xi)| \leq \|\varphi_i\|_{L^\infty} = \|\mathcal{F}\mathcal{F}^{-1}\varphi_i\|_{L^\infty} \leq \|\mathcal{F}^{-1}\varphi\|_{L^1} \leq C_{\mathcal{Q}, \Phi, 2} \quad (6.25)$$

for all $i \in I$ and $\xi \in \mathbb{R}^d$. Furthermore, as seen above, we have $\varphi_i \equiv 0$ for all $i \in I \setminus I_0$, with $I_0 \subset I$ countable. Next, for arbitrary $i \in I_0$, Plancherel’s theorem implies

$$\begin{aligned} \|\mathcal{F}^{-1}(\varphi_i \cdot f)\|_{L^2}^2 &= \|\varphi_i \cdot f\|_{L^2}^2 \\ (\text{since } \text{supp } \varphi_i \subset \mathcal{O}) &= w_i^{-2} \cdot \int_{\mathcal{O}} |w_i \varphi_i(\xi) \cdot f(\xi)|^2 d\xi \\ (\varphi_i \equiv 0 \text{ on } Q_i^c \text{ and } w_i \leq C w_0(\xi) \text{ on } Q_i) &\leq C^2 \cdot w_i^{-2} \cdot \int_{\mathcal{O}} |\varphi_i(\xi) \cdot (w_0 \cdot f)(\xi)|^2 d\xi. \end{aligned}$$

Multiplying with w_i and summing over $i \in I$ yields

$$\begin{aligned} \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^2, \ell_w^2)}^2 &= \sum_{i \in I} (w_i \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^2})^2 \\ &= \sum_{i \in I_0} (w_i \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^2})^2 \\ &\leq C^2 \cdot \sum_{i \in I_0} \int_{\mathcal{O}} |\varphi_i(\xi) \cdot (w_0 \cdot f)(\xi)|^2 d\xi \\ (\text{monotone convergence, } I_0 \text{ countable}) &= C^2 \cdot \int_{\mathcal{O}} \left(\sum_{i \in I_0} |\varphi_i(\xi)|^2 \right) \cdot |(w_0 \cdot f)(\xi)|^2 d\xi. \end{aligned}$$

But equation (6.25) yields for $\xi \in \mathcal{O}$ that

$$\sum_{i \in I_0} |\varphi_i(\xi)|^2 \leq \sum_{i \in I} |\varphi_i(\xi)|^2 \leq C_{\mathcal{Q}, \Phi, 2}^2 \cdot \sum_{i \in I} \mathbf{1}_{Q_i}(\xi) \leq N_{\mathcal{Q}} C_{\mathcal{Q}, \Phi, 2}^2,$$

where the last step used that \mathcal{Q} is admissible. All in all, we have thus shown

$$\begin{aligned} \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^2, \ell_w^2)}^2 &\leq C^2 \cdot \int_{\mathcal{O}} \left(\sum_{i \in I_0} |\varphi_i(\xi)|^2 \right) \cdot |(w_0 \cdot f)(\xi)|^2 d\xi \\ &\leq N_{\mathcal{Q}} \cdot (C C_{\mathcal{Q}, \Phi, 2})^2 \cdot \int_{\mathcal{O}} |(w_0 \cdot f)(\xi)|^2 d\xi \\ &= N_{\mathcal{Q}} \cdot (C C_{\mathcal{Q}, \Phi, 2})^2 \cdot \|f\|_{L_{w_0}^2(\mathcal{O})}^2 < \infty. \end{aligned}$$

We have thus shown $L_{w_0}^2(\mathcal{O}) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^2, \ell_w^2)$.

For the reverse inclusion, let $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^2, \ell_w^2) \subset \mathcal{D}'(\mathcal{O})$. Note that $L_{w_0}^2(\mathcal{O}) = L_{w_1}^2(\mathcal{O})$ if w_0, w_1 are measurable and equivalent, i.e. $w_0 \asymp w_1$. Furthermore, any two weights w_0, w_1 satisfying $w_\ell(\xi) \asymp w_i$ for $\xi \in Q_i$, $\ell \in \{0, 1\}$ and $i \in I$ are equivalent. Hence, we can assume without loss of generality that w_0 is as in equation (6.24). As seen above, this ensures $w_i \in C^\infty(\mathcal{O})$.

Thus, for $g \in C_c^\infty(\mathcal{O})$, we have $w_0 g \in C_c^\infty(\mathcal{O})$ as well. Now, note

$$\begin{aligned}
 |\langle f, w_0 g \rangle| &= \left| \sum_{i \in I} \langle \varphi_i f, w_0 g \rangle \right| \\
 (\varphi_i^* \equiv 1 \text{ on } Q_i \text{ and } \varphi_i \equiv 0 \text{ on } Q_i^c) &= \left| \sum_{i \in I} \langle \varphi_i f, \varphi_i^*(w_0 g) \rangle \right| \\
 &= \left| \sum_{i \in I} \langle \mathcal{F}^{-1}(\varphi_i f), \mathcal{F}[\varphi_i^*(w_0 g)] \rangle \right| \\
 &\leq \sum_{i \in I} w_i \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^2} w_i^{-1} \|\mathcal{F}[\varphi_i^*(w_0 g)]\|_{L^2} \\
 (\text{Cauchy-Schwarz and Plancherel}) &\leq \left[\sum_{i \in I} (w_i \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^2})^2 \right]^{1/2} \cdot \left[\sum_{i \in I} (w_i^{-1} \|\varphi_i^*(w_0 g)\|_{L^2})^2 \right]^{1/2}.
 \end{aligned}$$

The first factor in the last line is exactly $\|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^2, \ell_w^2)}$.

Now, let us consider the second factor: Because of $w_0 g \in C_c^\infty(\mathcal{O})$ and since $(\varphi_i)_{i \in I}$ is locally finite in \mathcal{O} , we have $\varphi_\ell \cdot (w_0 g) \equiv 0$ for all $\ell \in I \setminus F_0$, for some finite set $F_0 \subset I$. But $\varphi_i^* \cdot (w_0 g) \not\equiv 0$ implies $\varphi_\ell \cdot (w_0 g) \not\equiv 0$ for some $\ell \in i^*$ and hence $i \in \ell^* \subset F_0^* =: F$, with $F \subset I$ finite. Furthermore, for $\xi \in Q_i^*$, we have $\xi \in Q_\ell$ for some $\ell \in i^*$, so that we get $w_0(\xi) \leq C w_\ell \leq C C_{w, \mathcal{Q}} w_i$. Finally, as a consequence of equation (6.25), we have

$$|\varphi_i^*(\xi)| \leq \sum_{\ell \in i^*} |\varphi_\ell(\xi)| \leq C_{\mathcal{Q}, \Phi, 2} |i^*| \leq C_{\mathcal{Q}, \Phi, 2} N_{\mathcal{Q}}$$

for all $i \in I$ and $\xi \in \mathbb{R}^d$. Hence,

$$\begin{aligned}
 \sum_{i \in I} (w_i^{-1} \|\varphi_i^*(w_0 g)\|_{L^2})^2 &= \sum_{i \in F} \left(w_i^{-2} \cdot \int_{\mathbb{R}^d} |\varphi_i^*(\xi)|^2 |(w_0 g)(\xi)|^2 d\xi \right) \\
 (\text{since } g \in C_c^\infty(\mathcal{O})) &= \int_{\mathcal{O}} |g(\xi)|^2 \cdot \sum_{i \in F} |w_i^{-1} \cdot \varphi_i^*(\xi) w_0(\xi)|^2 d\xi,
 \end{aligned}$$

with

$$\begin{aligned}
 \sum_{i \in F} |w_i^{-1} \cdot \varphi_i^*(\xi) w_0(\xi)|^2 &\leq (C C_{w, \mathcal{Q}})^2 \cdot \sum_{i \in F} |\varphi_i^*(\xi)|^2 \\
 &\leq (C C_{w, \mathcal{Q}} C_{\mathcal{Q}, \Phi, 2} N_{\mathcal{Q}})^2 \cdot \sum_{i \in I} \mathbf{1}_{Q_i^*}(\xi) \\
 &\leq (C C_{w, \mathcal{Q}} C_{\mathcal{Q}, \Phi, 2} N_{\mathcal{Q}})^2 \cdot N_{\mathcal{Q}^*} \\
 (\text{cf. Lemma 2.9}) &\leq (C C_{w, \mathcal{Q}} C_{\mathcal{Q}, \Phi, 2} N_{\mathcal{Q}})^2 \cdot N_{\mathcal{Q}}^3.
 \end{aligned}$$

Putting everything together, we see

$$\begin{aligned}
 |\langle f, w_0 g \rangle| &\leq \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^2, \ell_w^2)} \cdot C C_{w, \mathcal{Q}} C_{\mathcal{Q}, \Phi, 2} N_{\mathcal{Q}}^{5/2} \cdot \sqrt{\int_{\mathcal{O}} |g(\xi)|^2 d\xi} \\
 &=: C_1 \cdot \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^2, \ell_w^2)} \cdot \|g\|_{L^2(\mathcal{O})}.
 \end{aligned}$$

Since this holds for all $g \in C_c^\infty(\mathcal{O})$ and since $C_c^\infty(\mathcal{O}) \leq L^2(\mathcal{O})$ is dense, the Riesz representation theorem for Hilbert spaces yields some $h \in L^2(\mathcal{O})$ with $\|h\|_{L^2} \leq C_1 \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^2, \ell_w^2)}$ and with

$$\langle f, w_0 g \rangle = \langle h, g \rangle = \langle w_0^{-1} h, w_0 g \rangle \quad \forall g \in C_c^\infty(\mathcal{O}).$$

Since $C_c^\infty(\mathcal{O}) \rightarrow C_c^\infty(\mathcal{O}), g \mapsto w_0 g$ is a bijection (because $w_0 : \mathcal{O} \rightarrow (0, \infty)$ is smooth), this means $f = w_0^{-1} h$ as elements of $\mathcal{D}'(\mathcal{O})$. Hence, $f = w_0^{-1} h \in L_{w_0}^2(\mathcal{O})$ with

$$\|f\|_{L_{w_0}^2} = \|w_0^{-1} h\|_{L_{w_0}^2} = \|h\|_{L^2} \leq C_1 \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^2, \ell_w^2)} < \infty.$$

For the final statement of the lemma, simply note that if $w_0, v_0 : \mathcal{O} \rightarrow (0, \infty)$ are the associated continuous weights for the discrete weights w, v and if $\xi \in \mathcal{O}$, then $\xi \in Q_i \cap P_j$ for suitable $i \in I$ and $j \in J$. Hence,

$$w_0(\xi) \asymp w_i \asymp v_j \asymp v_0(\xi),$$

so that we get $w_0 \asymp v_0$ and thus

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^2, \ell_w^2) = L_{w_0}^2(\mathcal{O}) = L_{v_0}^2(\mathcal{O}) = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^2, \ell_v^2),$$

with equivalent norms. \square

Finally, we consider the case in which \mathcal{Q}, \mathcal{P} are weakly equivalent. Again, we only consider the case of weighted Lebesgue sequence spaces as the global component. The main reason for this is that it is difficult to formulate what it means for two sequence spaces $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ to be “equivalent”. In the present case, we simply require the exponents q_1, q_2 of $\ell_w^{q_1}(I)$ and $\ell_v^{q_2}(J)$ to coincide, $q_1 = q_2$, and furthermore assume that $w_i \asymp v_j$ if $Q_i \cap P_j \neq \emptyset$.

Lemma 6.11. *Let $p, q \in (0, \infty]$ and let $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ be open. Assume that $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ are two L^p -decomposition coverings of \mathcal{O} . Let $w = (w_i)_{i \in I}$ be \mathcal{Q} -moderate and let $v = (v_j)_{j \in J}$ be \mathcal{P} -moderate. Assume that*

- *There is a constant $C_0 > 0$ with*

$$C_0^{-1} \cdot w_i \leq v_j \leq C_0 \cdot w_i \quad (6.26)$$

for all $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \emptyset$.

- *\mathcal{Q}, \mathcal{P} are weakly equivalent.*

Then we have $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, \ell_w^q) = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, \ell_v^q)$ with equivalent quasi-norms, as long as $p \in [1, \infty]$.

For $p \in (0, 1)$, the same holds under the following additional assumption: Because of $p \in (0, 1)$, the coverings $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ and $\mathcal{P} = (S_j P'_j + c_j)_{j \in J}$ are semi-structured. We assume that there is a constant $C_1 > 0$ such that one of the following two conditions holds:

- (1) *For arbitrary $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \emptyset$, we have*

$$\|T_i^{-1} S_j\| \leq C_1 \quad \text{and} \quad \|S_j^{-1} T_i\| \leq C_1.$$

- (2) *\mathcal{P} is almost subordinate to \mathcal{Q} and we have*

$$|\det(S_j^{-1} T_i)| \leq C_1$$

for all $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \emptyset$. \blacktriangleleft

Remark. (1) The two kinds of assumptions for $p \in (0, 1)$ seem to be somewhat artificial. But there is a more natural condition which implies the second condition. Indeed, assume that

- \mathcal{Q}, \mathcal{P} are (open) semi-structured coverings of \mathcal{O} ,
- \mathcal{Q} is tight,
- \mathcal{Q} and \mathcal{P} are weakly equivalent,
- all sets in \mathcal{Q} and \mathcal{P} are connected.

Then the sets in \mathcal{Q} and \mathcal{P} are in fact path-connected (as connected, open subsets of \mathbb{R}^d). Since the sets in \mathcal{Q} and \mathcal{P} are open, Corollary 2.13 shows that \mathcal{Q} and \mathcal{P} are equivalent, not just weakly equivalent. In particular, \mathcal{P} is almost subordinate to \mathcal{Q} , as required. Furthermore, there is $k \in \mathbb{N}_0$ and for each $i \in I$ some $j_i \in J$ with $Q_i \subset P_{j_i}^{k*}$. Hence, for $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \emptyset$, Lemma 2.11 yields $Q_i \subset P_j^{(2k+2)*}$. By tightness of \mathcal{Q} , this implies

$$\begin{aligned} |\det T_i| &\lesssim \lambda(Q_i) \leq \lambda(P_j^{(2k+2)*}) \\ &\leq \sum_{\ell \in j^{(2k+2)*}} \lambda(P_\ell) \\ &\lesssim \sum_{\ell \in j^{(2k+2)*}} |\det S_\ell| \\ \left(|j^{(2k+2)*}| \leq N_{\mathcal{P}}^{2k+2} \text{ and eq. (3.11)} \right) &\lesssim |\det S_j|, \end{aligned}$$

where all implied constants are independent of i, j . Thus, $|\det(S_j^{-1}T_i)| = |\det T_i| / |\det S_j| \lesssim 1$, so that the second additional assumption from the theorem is indeed satisfied.

(2) The proof shows that an alternative assumption for the case $p \in (0, 1)$ would be to assume

$$\lambda(\overline{Q_i} - \overline{P_j}) \lesssim \min\{|\det T_i|, |\det S_j|\} \quad \forall i \in I \text{ and } j \in J \text{ with } Q_i \cap P_j \neq \emptyset.$$

But this estimate is usually very hard to check without establishing one of the additional assumptions from the lemma above. \blacklozenge

Proof. Let $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_j)_{j \in J}$ be L^p -BAPUs for \mathcal{Q} and \mathcal{P} , respectively. As usual, let $J_i = \{j \in J \mid P_j \cap Q_i \neq \emptyset\}$ and define I_j analogously for $i \in I$ and $j \in J$. We first show $\varphi_i \psi_{J_i} = \varphi_i$. For $\xi \in \mathbb{R}^d$ with $\varphi_i(\xi) = 0$, this is clear. If otherwise $\varphi_i(\xi) \neq 0$, then $\xi \in Q_i \subset \mathcal{O}$ and each $j \in J$ with $\psi_j(\xi) \neq 0$ hence satisfies $\xi \in Q_i \cap P_j \neq \emptyset$, i.e. $j \in J_i$. Thus,

$$1 = \sum_{j \in J} \psi_j(\xi) = \sum_{j \in J_i} \psi_j(\xi) = \psi_{J_i}(\xi),$$

which implies $\varphi_i(\xi) \psi_{J_i}(\xi) = \varphi_i(\xi)$ also in this case. Note that the identity $\sum_{j \in J} \psi_j(\xi) = 1$ crucially used that \mathcal{Q}, \mathcal{P} both cover the *same* set \mathcal{O} . Analogously, one can show $\psi_j \varphi_{I_j} = \psi_j$.

We first prove the claim for $p \in [1, \infty]$. Here, it suffices to show $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, \ell_w^q)} \lesssim \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, \ell_v^q)}$, since the assumptions are symmetric in \mathcal{Q}, \mathcal{P} . Hence, let $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, \ell_v^q)$ be arbitrary. As seen above, we have $\varphi_i = \varphi_i \psi_{J_i}$. Together with Young's inequality $L^1 * L^p \hookrightarrow L^p$ and with the triangle inequality for L^p , this yields

$$\begin{aligned} w_i \|\mathcal{F}^{-1}[\varphi_i f]\|_{L^p} &= w_i \|\mathcal{F}^{-1}[\varphi_i \psi_{J_i} f]\|_{L^p} \\ &\leq \sum_{j \in J_i} w_i \|\mathcal{F}^{-1}[\varphi_i \psi_j f]\|_{L^p} \\ &\leq C_0 \sum_{j \in J_i} [v_j \|\mathcal{F}^{-1} \varphi_i\|_{L^1} \|\mathcal{F}^{-1}[\psi_j f]\|_{L^p}] \\ &\leq C_0 C_{\mathcal{Q}, \Phi, p} \cdot \sum_{j \in J_i} [v_j \|\mathcal{F}^{-1}[\psi_j f]\|_{L^p}] \end{aligned} \tag{6.27}$$

for all $i \in I$.

In case of $q = \infty$, this implies

$$\begin{aligned} \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, \ell_w^q)} &= \sup_{i \in I} w_i \|\mathcal{F}^{-1}[\varphi_i f]\|_{L^p} \\ &\leq \sup_{i \in I} C_0 C_{\mathcal{Q}, \Phi, p} \cdot |J_i| \cdot \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, \ell_v^q)} \\ &\leq C_0 C_{\mathcal{Q}, \Phi, p} \cdot N(\mathcal{Q}, \mathcal{P}) \cdot \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, \ell_v^q)} < \infty. \end{aligned}$$

For $q \in (0, \infty)$, we use the uniform bound $|J_i| \leq N(\mathcal{Q}, \mathcal{P}) =: N$ to obtain $C_2 = C_2(N, q) > 0$ with

$$\left(\sum_{j \in J_i} [v_j \|\mathcal{F}^{-1}[\psi_j f]\|_{L^p}] \right)^q \leq C_2 \cdot \sum_{j \in J_i} [v_j \|\mathcal{F}^{-1}[\psi_j f]\|_{L^p}]^q$$

for all $i \in I$. This implies

$$\begin{aligned} \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, \ell_w^q)}^q &= \sum_{i \in I} [w_i \|\mathcal{F}^{-1}[\varphi_i f]\|_{L^p}]^q \\ &\leq (C_0 C_{\mathcal{Q}, \Phi, p})^q C_2 \cdot \sum_{i \in I} \sum_{j \in J_i} [v_j \|\mathcal{F}^{-1}[\psi_j f]\|_{L^p}]^q \\ &= (C_0 C_{\mathcal{Q}, \Phi, p})^q C_2 \cdot \sum_{j \in J} \sum_{i \in I_j} [v_j \|\mathcal{F}^{-1}[\psi_j f]\|_{L^p}]^q \\ &\leq (C_0 C_{\mathcal{Q}, \Phi, p})^q C_2 \cdot N(\mathcal{P}, \mathcal{Q}) \cdot \sum_{j \in J} [v_j \|\mathcal{F}^{-1}[\psi_j f]\|_{L^p}]^q \\ &= (C_0 C_{\mathcal{Q}, \Phi, p})^q C_2 \cdot N(\mathcal{P}, \mathcal{Q}) \cdot \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, \ell_v^q)}^q < \infty. \end{aligned}$$

Here, we used the equivalence $i \in I_j \Leftrightarrow j \in J_i$, as well as the estimate $|I_j| \leq N(\mathcal{P}, \mathcal{Q})$ for all $j \in J$.

Now, we consider the case $p \in (0, 1)$. Here, we first show that both of the additional assumptions imply that there are constants $C_3, R > 0$ such that

$$|\det(S_j^{-1}T_i)| \leq C_3 \quad (6.28)$$

and

$$P_j \subset T_i [\overline{B_R}(0)] + b_i \quad (6.29)$$

hold for all $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \emptyset$.

The inequality concerning the determinant is clear (with $C_3 = C_1$) in case of the second assumption. In case of the first assumption, we use Hadamard's inequality $|\det A| \leq \|A\|^d$ for arbitrary matrices $A \in \mathbb{R}^{d \times d}$ to deduce

$$|\det(S_j^{-1}T_i)| \leq \|S_j^{-1}T_i\|^d \leq C_1^d$$

for all $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \emptyset$.

For the inclusion $P_j \subset T_i (\overline{B_R}(0)) + b_i$, let us first consider the case of the second assumption. Since \mathcal{P} is almost subordinate to \mathcal{Q} , the constant $k := k(\mathcal{P}, \mathcal{Q}) \in \mathbb{N}_0$ is well-defined and we have $P_j \subset Q_{i_j}^{k*}$ for all $j \in J$ and suitable $i_j \in I$. In case of $Q_i \cap P_j \neq \emptyset$, this implies $P_j \subset Q_i^{(2k+2)*}$ by Lemma 2.11. But finally, Lemma 2.7 yields

$$P_j \subset Q_i^{(2k+2)*} \subset T_i [\overline{B_{(2C_{\mathcal{Q}}+1)^{2k+2}R_{\mathcal{Q}}}}(0)] + b_i$$

for all $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \emptyset$.

In case of the first assumption, define $R_0 := \max\{R_{\mathcal{Q}}, R_{\mathcal{P}}\}$ and let $\xi \in Q_i \cap P_j \neq \emptyset$. This yields $\omega \in Q'_i \subset \overline{B_{R_0}}(0)$ and $\eta \in P'_j \subset \overline{B_{R_0}}(0)$ with

$$\xi = T_i \omega + b_i = S_j \eta + c_j.$$

Hence, $b_i = S_j \eta + c_j - T_i \omega$. Now, let $x \in \overline{B_{R_0}}(0)$ be arbitrary. We have

$$\begin{aligned} |T_i^{-1}(S_j x + c_j - b_i)| &= |T_i^{-1}(S_j x + c_j - (S_j \eta + c_j - T_i \omega))| \\ &\leq |T_i^{-1}S_j x| + |T_i^{-1}S_j \eta| + |\omega| \\ &\leq C_1 |x| + C_1 |\eta| + |\omega| \\ &\leq R_0 (1 + 2C_1) =: R \end{aligned}$$

and hence

$$S_j x + c_j \in T_i [\overline{B_R}(0)] + b_i.$$

Since $x \in \overline{B_{R_0}}(0)$ was arbitrary, this implies

$$P_j = S_j P'_j + c_j \subset S_j [\overline{B_{R_0}}(0)] + c_j \subset T_i [\overline{B_R}(0)] + b_i$$

for all $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \emptyset$.

With this preparation, we can prove the actual claim: The argument is almost the same as for $p \in [1, \infty]$; only the application of Young's inequality and of the triangle inequality for L^p in equation (6.27) need to be replaced. In short, we only need to establish equation (6.27) also for $p \in (0, 1)$; the rest of the proof then proceeds as for $p \in [1, \infty]$. The only caveat is that since the assumptions are *not* symmetric in \mathcal{Q}, \mathcal{P} anymore, we also need to show the “reverse” version of equation (6.27).

Since $\|\cdot\|_{L^p}$ is a quasi-norm and because of the uniform bound $|J_i| \leq N(\mathcal{Q}, \mathcal{P}) = N$, there is a constant $C_4 = C_4(N, p) > 0$ with

$$\begin{aligned} w_i \|\mathcal{F}^{-1}[\varphi_i f]\|_{L^p} &= w_i \|\mathcal{F}^{-1}[\varphi_i \psi_{J_i} f]\|_{L^p} \\ &\leq C_4 \cdot w_i \cdot \sum_{j \in J_i} \|\mathcal{F}^{-1}[\varphi_i \psi_j f]\|_{L^p} \\ &\stackrel{(\dagger)}{\leq} C_4 C_5 \cdot w_i \cdot \sum_{j \in J_i} |\det T_i|^{\frac{1}{p}-1} \|\mathcal{F}^{-1}\varphi_i\|_{L^p} \|\mathcal{F}^{-1}[\psi_j f]\|_{L^p} \\ &\leq C_0 C_4 C_5 C_{\mathcal{Q}, \Phi, p} \cdot \sum_{j \in J_i} v_j \|\mathcal{F}^{-1}[\psi_j f]\|_{L^p} \end{aligned} \quad (6.30)$$

for all $i \in I$. Using this estimate instead of equation (6.27), the proof for $p \in (0, 1)$ can be completed just as for $p \in [1, \infty]$.

In equation (6.30), the step marked with (\dagger) is justified as follows: By possibly enlarging the constant R from equation (6.29), we can assume $R \geq R_0 = \max\{R_Q, R_P\}$. This yields the inclusions $\text{supp } \varphi_i \subset \overline{Q_i} \subset T_i [\overline{B_R}(0)] + b_i =: M_i$ and

$$\text{supp } \psi_j \subset \overline{P_j} \subset T_j [\overline{B_R}(0)] + b_j =: M_j$$

for all $j \in J$, cf. equation (6.29). Thus, Theorem 3.4 shows

$$\begin{aligned} \|\mathcal{F}^{-1}[\varphi_i \psi_j f]\|_{L^p} &\leq [\lambda(M_i - M_j)]^{\frac{1}{p}-1} \|\mathcal{F}^{-1}\varphi_i\|_{L^p} \|\mathcal{F}^{-1}[\psi_j f]\|_{L^p} \\ &= [\lambda(T_i[\overline{B_R}(0)] - \overline{B_R}(0))]^{\frac{1}{p}-1} \|\mathcal{F}^{-1}\varphi_i\|_{L^p} \|\mathcal{F}^{-1}[\psi_j f]\|_{L^p} \\ &\leq [\lambda(\overline{B_{2R}}(0))]^{\frac{1}{p}-1} \cdot |\det T_i|^{\frac{1}{p}-1} \|\mathcal{F}^{-1}\varphi_i\|_{L^p} \|\mathcal{F}^{-1}[\psi_j f]\|_{L^p} \\ &=: C_5 \cdot |\det T_i|^{\frac{1}{p}-1} \|\mathcal{F}^{-1}\varphi_i\|_{L^p} \|\mathcal{F}^{-1}[\psi_j f]\|_{L^p}, \end{aligned} \quad (6.31)$$

where the constant C_5 only depends on $R > 0$, on $d \in \mathbb{N}$ and on $p \in (0, 1)$.

For the “reverse” version of equation (6.27), a similar argument applies: Because of the uniform bound $|I_j| \leq N(\mathcal{P}, \mathcal{Q}) =: N_2$, there is a constant $C_6 = C_6(N_2, p) > 0$ with

$$\begin{aligned} v_j \|\mathcal{F}^{-1}[\psi_j f]\|_{L^p} &= v_j \|\mathcal{F}^{-1}[\psi_j \varphi_{I_j} f]\|_{L^p} \\ &\leq C_6 \cdot v_j \cdot \sum_{i \in I_j} \|\mathcal{F}^{-1}[\psi_j \varphi_i f]\|_{L^p} \\ &\stackrel{(\dagger)}{\leq} C_5 C_6 \cdot v_j \cdot \sum_{i \in I_j} |\det T_i|^{\frac{1}{p}-1} \|\mathcal{F}^{-1}\psi_j\|_{L^p} \|\mathcal{F}^{-1}[\varphi_i f]\|_{L^p} \\ &\stackrel{(\ddagger)}{\leq} C_0 C_3^{\frac{1}{p}-1} C_5 C_6 \cdot \sum_{i \in I_j} \left[|\det S_j|^{\frac{1}{p}-1} \|\mathcal{F}^{-1}\psi_j\|_{L^p} \cdot w_i \|\mathcal{F}^{-1}[\varphi_i f]\|_{L^p} \right] \\ &\leq C_0 C_3^{\frac{1}{p}-1} C_5 C_6 C_{\mathcal{P}, \Psi, p} \cdot \sum_{i \in I_j} [w_i \|\mathcal{F}^{-1}[\varphi_i f]\|_{L^p}]. \end{aligned}$$

Here, we used $\frac{1}{p} - 1 > 0$, together with estimate (6.28) at (\ddagger) . The justification for (\dagger) is exactly as in equation (6.31) above. \square

6.3. Improved necessary conditions. In this subsection, we use many of the techniques from the preceding subsection to improve upon the “elementary” necessary conditions that we developed in Subsection 6.1.

As in the previous subsection, we will use functions of the form

$$f = \sum_{i \in I_0} M_{z_i}(\varepsilon_i c_i \cdot \gamma_i),$$

with $\gamma_i \in C_c^\infty(Q_i)$, to “test” the embedding $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$. Our next lemma indicates how we will choose the functions γ_i .

Lemma 6.12. *Assume that $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ is a semi-structured covering of the open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ with L^p -BAPU $\Phi = (\varphi_i)_{i \in I}$ for some $p \in (0, \infty]$.*

Let $q \in [\min\{1, p\}, \infty]$ and $\ell \in \mathbb{N}_0$. Then there is a constant

$$C_1 = C_1(d, p, q, \ell, \mathcal{Q}, C_{\mathcal{Q}, \Phi, p}) > 0$$

such that

$$\|\mathcal{F}^{-1}\varphi_{M_i}\|_{L^q} \leq C_1 \cdot |\det T_i|^{1-\frac{1}{q}} \quad (6.32)$$

holds for all $i \in I$ and all sets $M_i \subset i^{\ell}$.*

If \mathcal{Q} is tight, there are two families of functions $(\gamma_i)_{i \in I}, (\phi_i)_{i \in I}$ with the following properties:

- (1) *We have $\gamma_i, \phi_i \in C_c^\infty(\mathcal{O})$ and $\text{supp } \gamma_i, \text{supp } \phi_i \subset Q_i$, as well as $\gamma_i, \phi_i \geq 0$ for all $i \in I$.*
- (2) *We have $\phi_i \gamma_i = \gamma_i$ for all $i \in I$.*

(3) For each $s \in (0, \infty]$, there are constants $C_2 = C_2(d, s) > 0$ and $C_3 = C_3(d, s) > 0$ with

$$\|\mathcal{F}^{-1}\gamma_i\|_{L^s} = C_2 \cdot \varepsilon_{\mathcal{Q}}^{d(1-\frac{1}{s})} \cdot |\det T_i|^{1-\frac{1}{s}} \quad \text{and} \quad \|\mathcal{F}^{-1}\phi_i\|_{L^s} = C_3 \cdot \varepsilon_{\mathcal{Q}}^{d(1-\frac{1}{s})} \cdot |\det T_i|^{1-\frac{1}{s}}$$

for all $i \in I$. ◀

Proof. Let $r := \min\{1, p\}$. By definition of an L^p -BAPU, we know that

$$C_{\mathcal{Q}, \Phi, p} = C_{\mathcal{Q}, \Phi, r} = \sup_{i \in I} |\det T_i|^{\frac{1}{r}-1} \|\mathcal{F}^{-1}\varphi_i\|_{L^r}$$

is finite. By applying Lemma 5.3 (with $k = 0$, $p_0 = p_1 = r$ and $p_2 = q \geq r = p_1$, as well as $f \equiv 1$), we get a constant $K_0 = K_0(d, r, \mathcal{Q}) = K_0(d, p, \mathcal{Q}) > 0$ with

$$\begin{aligned} \|\mathcal{F}^{-1}\varphi_i\|_{L^q} &\leq K_0 \cdot |\det T_i|^{\frac{1}{r}-\frac{1}{q}} \|\mathcal{F}^{-1}\varphi_i\|_{L^r} \\ &\leq K_0 C_{\mathcal{Q}, \Phi, p} |\det T_i|^{\frac{1}{r}-\frac{1}{q}} |\det T_i|^{1-\frac{1}{r}} \\ &= K_0 C_{\mathcal{Q}, \Phi, p} |\det T_i|^{1-\frac{1}{q}} \end{aligned}$$

for all $i \in I$, because of $\varphi_i \equiv 0$ on $\mathcal{O} \setminus Q_i$.

Since $\|\cdot\|_{L^q}$ is a quasi-norm and because of the uniform bound $|M_i| \leq |i^{\ell*}| \leq N_{\mathcal{Q}}^{\ell}$ (which was shown in Lemma 2.9), we obtain a constant $K_1 = K_1(N_{\mathcal{Q}}, \ell, q) > 0$ with

$$\begin{aligned} \|\mathcal{F}^{-1}\varphi_{M_i}\|_{L^q} &\leq K_1 \cdot \sum_{j \in M_i} \|\mathcal{F}^{-1}\varphi_j\|_{L^q} \\ &\leq K_0 K_1 C_{\mathcal{Q}, \Phi, p} \cdot \sum_{j \in M_i} |\det T_j|^{1-\frac{1}{q}} \end{aligned}$$

for all $i \in I$ and each subset $M_i \subset i^{\ell*}$. Using Lemma 2.9 and the \mathcal{Q} -moderateness of the weight $(|\det T_i|)_{i \in I}$ (with “moderateness constant” $C_{\mathcal{Q}}^d$, cf. equation (3.11)), it is easy to see that there is a constant $K_2 = K_2(C_{\mathcal{Q}}, d, \ell, q) \geq 1$ with

$$K_2^{-1} |\det T_i|^{1-\frac{1}{q}} \leq |\det T_j|^{1-\frac{1}{q}} \leq K_2 |\det T_i|^{1-\frac{1}{q}}$$

for all $i \in I$ and $j \in i^{\ell*} \supset M_i$.

All in all, we conclude

$$\begin{aligned} \|\mathcal{F}^{-1}\varphi_{M_i}\|_{L^q} &\leq K_0 K_1 K_2 C_{\mathcal{Q}, \Phi, p} \cdot \sum_{j \in M_i} |\det T_i|^{1-\frac{1}{q}} \\ &\leq K_0 K_1 K_2 N_{\mathcal{Q}}^{\ell} C_{\mathcal{Q}, \Phi, p} \cdot |\det T_i|^{1-\frac{1}{q}} \end{aligned}$$

for all $i \in I$ and each subset $M_i \subset i^{\ell*}$. Here, we again used the bound $|M_i| \leq |i^{\ell*}| \leq N_{\mathcal{Q}}^{\ell}$ in the last step. We have thus established equation (6.32), for $C_1 := K_0 K_1 K_2 N_{\mathcal{Q}}^{\ell} C_{\mathcal{Q}, \Phi, p}$.

For the construction of $(\gamma_i)_{i \in I}, (\phi_i)_{i \in I}$, fix a non-trivial nonnegative function $\gamma \in C_c^{\infty}(B_{1/2}(0))$ and a nonnegative function $\phi \in C_c^{\infty}(B_1(0))$ with $\phi \equiv 1$ on $B_{1/2}(0)$ and let $\varepsilon := \varepsilon_{\mathcal{Q}}$. For each $i \in I$, choose $c_i \in \mathbb{R}^d$ with $B_{\varepsilon}(c_i) \subset Q'_i$ (cf. Remark 2.6 for the existence of c_i) and define

$$\gamma_i := L_{b_i} [(L_{c_i} [\gamma \circ \varepsilon^{-1} \text{id}]) \circ T_i^{-1}] \quad \text{and} \quad \phi_i := L_{b_i} [(L_{c_i} [\phi \circ \varepsilon^{-1} \text{id}]) \circ T_i^{-1}]$$

Clearly, $\gamma_i \geq 0$ and $\phi_i \geq 0$. Furthermore, $\text{supp } \gamma \subset \text{supp } \phi$ and hence

$$\begin{aligned} \text{supp } \gamma_i &\subset \text{supp } \phi_i = \text{supp} ((L_{c_i} [\phi \circ \varepsilon^{-1} \text{id}]) \circ T_i^{-1}) + b_i \\ &= T_i (\text{supp} (L_{c_i} [\phi \circ \varepsilon^{-1} \text{id}])) + b_i \\ &= T_i (\text{supp} (\phi \circ \varepsilon^{-1} \text{id}) + c_i) + b_i \\ &\subset T_i [B_{\varepsilon}(c_i)] + b_i \\ &\subset T_i Q'_i + b_i = Q_i \subset \mathcal{O}, \end{aligned}$$

and thus $\gamma_i, \phi_i \in C_c^{\infty}(\mathcal{O})$. Finally, since $\phi \equiv 1$ on $B_{1/2}(0) \supset \text{supp } \gamma$, it is easy to see $\phi\gamma = \gamma$, which then implies $\phi_i\gamma_i = \gamma_i$, as desired.

Finally, a calculation using standard properties of the Fourier transform (cf. [8, Theorem 8.22]) shows

$$(\mathcal{F}^{-1}\gamma_i)(x) = \varepsilon^d \cdot |\det T_i| \cdot M_{b_i} [(M_{c_i} [(\mathcal{F}^{-1}\gamma) \circ \text{id}]) \circ T_i^T]$$

and thus

$$\begin{aligned} \|\mathcal{F}^{-1}\gamma_i\|_{L^s} &= \varepsilon^d \cdot |\det T_i| \cdot |\det T_i|^{-\frac{1}{s}} \cdot \|(\mathcal{F}^{-1}\gamma) \circ \text{id}\|_{L^s} \\ &= \|\mathcal{F}^{-1}\gamma\|_{L^s} \cdot |\det T_i|^{1-\frac{1}{s}} \cdot \varepsilon^{d(1-\frac{1}{s})} \end{aligned}$$

for all $s \in (0, \infty]$. Hence, the family $(\gamma_i)_{i \in I}$ has the desired properties, with $C_2(d, s) = \|\mathcal{F}^{-1}\gamma\|_{L^s}$, where γ depends only on d . Exactly the same calculation shows

$$\|\mathcal{F}^{-1}\phi_i\|_{L^s} = C_3(d, s) \cdot \varepsilon^{d(1-\frac{1}{s})} \cdot |\det T_i|^{1-\frac{1}{s}}$$

for all $i \in I$, with $C_3(d, s) = \|\mathcal{F}^{-1}\phi\|_{L^s}$. \square

Given these preparations, we can finally state and prove the first main result of this subsection. A remarkable property of the following result is—at least for $p_1 \in [1, \infty]$ —that we need to impose virtually no restrictions upon the relation between the coverings \mathcal{Q} and \mathcal{P} —only that $P_j \subset \mathcal{O}$ for all $j \in J_0$. Only for $p_1 \in (0, 1)$, we assume that \mathcal{P}_{J_0} is almost subordinate to \mathcal{Q} . We remark that the following theorem is an improved version of (the second part of) [23, Theorem 5.3.6] from my PhD thesis.

Theorem 6.13. Let $\emptyset \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p_1, p_2 \in (0, \infty]$ and let $\mathcal{P} = (P_j)_{j \in J} = (S_j P'_j + c_j)_{j \in J}$ be a *tight* semi-structured L^{p_2} -decomposition covering of \mathcal{O}' . Furthermore, let $\mathcal{Q} = (Q_i)_{i \in I}$ be an L^{p_1} -decomposition covering of \mathcal{O} . Finally, let $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ be \mathcal{Q} -regular and \mathcal{P} -regular with triangle constants $C_Y, C_Z \geq 1$, respectively.

Let $J_0 \subset J$ be arbitrary with $P_j \subset \mathcal{O}$ for all $j \in J_0$. Define

$$K := \bigcup_{j \in J_0} P_j \subset \mathcal{O} \cap \mathcal{O}'$$

and assume that there is a bounded linear map

$$\iota : \left(\mathcal{D}_K^{\mathcal{Q}, p_1, Y}, \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)} \right) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z) \quad (6.33)$$

satisfying $\langle \iota f, \varphi \rangle = \langle f, \varphi \rangle$ for all $\varphi \in C_c^\infty(\mathcal{O} \cap \mathcal{O}')$ and all $f \in \mathcal{D}_K^{\mathcal{Q}, p_1, Y}$. In case of $p_1 \in (0, 1)$, assume additionally that $\mathcal{P}_{J_0} := (P_j)_{j \in J_0}$ is almost subordinate to \mathcal{Q} , i.e., there is $k \in \mathbb{N}_0$ such that for each $j \in J_0$ the inclusion $P_j \subset Q_{i_j}^{k*}$ is valid for a suitable $i_j \in I$.

Then the embedding

$$\eta : \ell_0(J_0) \cap Y \left([\ell^{p_1}(J_i \cap J_0)]_{i \in I} \right) \hookrightarrow Z_{|\det S_j|^{p_1^{-1} - p_2^{-1}}}, \quad (6.34)$$

$$(x_j)_{j \in J_0} \mapsto (x_j)_{j \in J} \text{ with } x_j = 0 \text{ for } j \in J \setminus J_0$$

is well-defined and bounded, with

$$\|\eta\| \leq C \cdot \|\iota\|$$

for some constant

$$C = C(d, p_1, p_2, k, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, C_Z, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}),$$

where the dependence on k can be dropped for $p_1 \in [1, \infty]$. Here, the L^{p_1} -BAPU $\Phi = (\varphi_i)_{i \in I}$ has to be used to calculate the (quasi)-norm on the decomposition space $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$ when calculating $\|\iota\|$.

If Z satisfies the Fatou property, the embedding η is bounded (with the same estimate for the norm) even *without* restricting to $\ell_0(J_0)$. \blacktriangleleft

Remark 6.14. Two remarks are in order:

1. As observed above, the preceding theorem imposes essentially no restriction upon the relation between \mathcal{Q}, \mathcal{P} —at least for $p_1 \in [1, \infty]$. Nevertheless, under these minimal assumptions, the theorem is only of limited value since it is difficult to (dis)prove boundedness of the embedding (6.34). In contrast, if \mathcal{P}_{J_0} is almost subordinate to \mathcal{Q} and if the “global” components Y, Z are in fact weighted

Lebesgue sequence spaces, one can use Corollary 5.12 to greatly simplify the task of checking whether the embedding (6.34) is bounded; see also Theorem 7.2 below.

2. Despite the pessimistic view of the preceding point, one can still derive nontrivial conditions from boundedness of the embedding (6.34): Indeed, let $j_0 \in J_0$ and $i \in I$ with $Q_i \cap P_{j_0} \neq \emptyset$ and assume $\mathbf{1}_{I_{j_0}} \in Y$. Now, for $\ell \in I$ there are two cases:

Case 1. We have $j_0 \in J_\ell \cap J_0$. This implies $\ell \in I_{j_0}$ and thus $\|\delta_{j_0}\|_{\ell^{p_1}(J_\ell \cap J_0)} = 1 \leq \mathbf{1}_{I_{j_0}}(\ell)$.

Case 2. We have $j_0 \notin J_\ell \cap J_0$. This implies $\|\delta_{j_0}\|_{\ell^{p_1}(J_\ell \cap J_0)} = 0 \leq \mathbf{1}_{I_{j_0}}(\ell)$.

All in all, we get $\delta_{j_0} \in \ell_0(J_0) \cap Y([\ell^{p_1}(J_\ell \cap J_0)]_{\ell \in I})$ with

$$\begin{aligned} |\det S_{j_0}|^{p_1^{-1}-p_2^{-1}} \cdot \|\delta_{j_0}\|_Z &= \|\delta_{j_0}\|_{(Z|_{J_0})|_{\det S_j|^{p_1^{-1}-p_2^{-1}}}} \\ &\leq \|\eta\| \cdot \|\delta_{j_0}\|_{Y([\ell^{p_1}(J_\ell \cap J_0)]_{\ell \in I})} \\ &= \|\eta\| \cdot \left\| \left(\|\delta_{j_0}\|_{\ell^{p_1}(J_\ell \cap J_0)} \right)_{\ell \in I} \right\|_Y \\ &\leq \|\eta\| \cdot \|\mathbf{1}_{I_{j_0}}\|_Y. \end{aligned}$$

Of course, in general, it is not easy to guarantee $\mathbf{1}_{I_{j_0}} \in Y$, or to estimate $\|\mathbf{1}_{I_{j_0}}\|_Y$. But if \mathcal{P}_{J_0} is almost subordinate to \mathcal{P} , the above estimate can be improved: Indeed, simply assume $\delta_i \in Y$ (and still $Q_i \cap P_{j_0} \neq \emptyset$). Lemma 2.11 shows $I_{j_0} \subset i^{(2k+2)*}$ and hence $\mathbf{1}_{I_{j_0}} \leq \Theta_{2k+2}\delta_i$. But in view of Lemma 3.9 and since Y is solid, this implies $\mathbf{1}_{I_{j_0}} \in Y$ with $\|\mathbf{1}_{I_{j_0}}\|_Y \leq \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}^{2k+2} \cdot \|\delta_i\|_Y$. Hence, if \mathcal{P}_{J_0} is almost subordinate to \mathcal{Q} , then

$$|\det S_{j_0}|^{p_1^{-1}-p_2^{-1}} \cdot \|\delta_{j_0}\|_Z \lesssim \|\delta_i\|_Y \quad \forall j_0 \in J_0 \text{ and } i \in I \text{ with } Q_i \cap P_{j_0} \neq \emptyset \text{ and } \delta_i \in Y. \quad (6.35)$$

This estimate can be seen as a generalization to the case $p_1 \neq p_2$ of the estimate $\|\delta_j\|_Z \lesssim \|\delta_i\|_Y$ from Lemma 6.1, which required $Q_i \cap P_j \neq \emptyset$ and $p_1 = p_2$. In many cases, this—very simple—estimate already suffices to show that a certain embedding between decomposition spaces can *not* exist. ♦

Proof of Theorem 6.13. For $j \in J_0$, we have $P_j \subset \mathcal{O}$ and $P_j \neq \emptyset$ (by tightness of \mathcal{P}) and hence $P_j \cap Q_i \neq \emptyset$ for some $i \in I$, since \mathcal{Q} covers \mathcal{O} . Hence, $j \in J_i \cap J_0$ for some $i \in I$. All in all, this shows $J_0 = \bigcup_{i \in I} (J_i \cap J_0)$, so that

$$V := Y([\ell^{p_1}(J_i \cap J_0)]_{i \in I}) \leq \mathbb{C}^{J_0} \quad \text{and} \quad V_0 := \ell_0(J_0) \cap V$$

are solid sequence spaces on J_0 .

We begin by setting up some quantities which we will need for constructing suitable functions to “test” the embedding ι . For $r_0 := N_{\mathcal{P}}^3$, the disjointization lemma (Lemma 2.14) yields a partition $J = \bigsqcup_{r=1}^{r_0} J^{(r)}$ satisfying $P_j^* \cap P_\ell^* = \emptyset$ for all $j, \ell \in J^{(r)}$ with $j \neq \ell$ and arbitrary $r \in \underline{r_0}$. Let $\varepsilon := \varepsilon_{\mathcal{P}}$. With this notation, Lemma 6.12 (applied to the tight semi-structured covering \mathcal{P}) yields two families $(\gamma_j)_{j \in J}, (\phi_j)_{j \in J}$ of nonnegative functions $\gamma_j, \phi_j \in C_c^\infty(\mathcal{O})$ with $\phi_j \gamma_j = \gamma_j$ and $\text{supp } \gamma_j, \text{supp } \phi_j \subset P_j$, as well as

$$\|\mathcal{F}^{-1}\gamma_j\|_{L^p} = C_1^{(p)} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot |\det S_j|^{1-\frac{1}{p}} \quad \text{and} \quad \|\mathcal{F}^{-1}\phi_j\|_{L^p} = C_2^{(p)} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot |\det S_j|^{1-\frac{1}{p}}$$

for all $j \in J$ and $p \in (0, \infty]$, with $C_1^{(p)} = C_1^{(p)}(d)$ and $C_2^{(p)} = C_2^{(p)}(d)$. Note that

$$\text{supp } \gamma_j, \text{supp } \phi_j \subset P_j \subset K \subset \mathcal{O} \quad \forall j \in J_0.$$

Let $\Psi = (\psi_j)_{j \in J}$ be an L^{p_2} -BAPU for \mathcal{P} .

Now, let $c = (c_j)_{j \in J_0} \in V_0$ be arbitrary and let $M := \text{supp } c$, which is a finite subset of J_0 . Define $\zeta_j := |\det S_j|^{p_1^{-1}-1} \cdot c_j$ for $j \in J_0$ and note $\text{supp } \zeta = \text{supp } c = M$ for $\zeta = (\zeta_j)_{j \in J_0}$. Now, for $z = (z_j)_{j \in J_0} \in (\mathbb{R}^d)^{J_0}$ and $r \in \underline{r_0}$, define $J_0^{(r)} := J_0 \cap J^{(r)}$, as well as $M^{(r)} := M \cap J^{(r)}$ and

$$f_z^{(r)} := \sum_{j \in J_0^{(r)}} M_{z_j}(\zeta_j \cdot \gamma_j) = \sum_{j \in M^{(r)}} M_{z_j}(\zeta_j \cdot \gamma_j). \quad (6.36)$$

Note that we have $\text{supp } \gamma_j \subset K \subset \mathcal{O}$ for all $j \in J_0$, so that we have $f_z^{(r)} \in \mathcal{D}_K^{\mathcal{Q}, p_1, Y}$ if we can show $f_z^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$, at least for suitable values of z .

We will now show that this is indeed true. To this end, first note for $i \in I$ that

$$\begin{aligned} \varphi_i \cdot f_z^{(r)} &= \sum_{j \in J_0^{(r)}} [\zeta_j \cdot M_{z_j}(\varphi_i \cdot \gamma_j)] \\ (\varphi_i \gamma_j \neq 0 \text{ only for } Q_i \cap P_j \neq \emptyset) &= \sum_{j \in J_i \cap J_0^{(r)}} [\zeta_j \cdot M_{z_j}(\varphi_i \cdot \gamma_j)] \\ &= \varphi_i \cdot \sum_{j \in J_i \cap J_0^{(r)}} [M_{z_j}(\zeta_j \cdot \gamma_j)] \\ &=: \varphi_i \cdot F_i^{(r, z)}. \end{aligned}$$

Now, we invoke Corollary 6.6 (with $f_j = \zeta_j \gamma_j \in \mathcal{S}(\mathbb{R}^d)$ for $j \in M$) to obtain a family of modulations $(z_j)_{j \in M}$ (depending on the chosen f_j and thus on ζ_j , as well as on p_1) such that we have for every subset $S \subset M$ the estimate

$$\begin{aligned} \left\| \mathcal{F}^{-1} \left[\sum_{j \in S} M_{z_j}(\zeta_j \gamma_j) \right] \right\|_{L^{p_1}} &\leq 2 \left\| (\|\mathcal{F}^{-1}(\zeta_j \gamma_j)\|_{L^{p_1}})_{j \in S} \right\|_{\ell^{p_1}} \\ &= 2C_1^{(p_1)} \varepsilon^{d(1-p_1^{-1})} \cdot \left\| (\zeta_j \cdot |\det S_j|^{1-p_1^{-1}})_{j \in S} \right\|_{\ell^{p_1}} \\ &= 2C_1^{(p_1)} \varepsilon^{d(1-p_1^{-1})} \cdot \left\| (c_j)_{j \in S} \right\|_{\ell^{p_1}}. \end{aligned}$$

In particular, if we set (e.g.) $z_j = 0$ for $j \in J \setminus M$, we get

$$\begin{aligned} \left\| \mathcal{F}^{-1} F_i^{(r, z)} \right\|_{L^{p_1}} &= \left\| \mathcal{F}^{-1} \left[\sum_{j \in J_i \cap M^{(r)}} M_{z_j}(\zeta_j \gamma_j) \right] \right\|_{L^{p_1}} \\ &\leq C_2 \cdot \left\| (c_j)_{j \in J_i \cap M^{(r)}} \right\|_{\ell^{p_1}} \\ &\leq C_2 \cdot \left\| (c_j)_{j \in J_i \cap J_0} \right\|_{\ell^{p_1}} = C_2 \cdot \mu_i \end{aligned}$$

for all $i \in I$, with $C_2 := 2C_1^{(p_1)} \varepsilon^{d(1-p_1^{-1})}$ and $\mu_i := \left\| (c_j)_{j \in J_i \cap J_0} \right\|_{\ell^{p_1}}$. Observe that $\mu := (\mu_i)_{i \in I} \in Y$ with $\|\mu\|_Y = \|c\|_V$, because of $c \in V = Y \left([\ell^{p_1} (J_i \cap J_0)]_{i \in I} \right)$.

Now, as usual, there are two cases for p_1 :

Case 1. $p_1 \in [1, \infty]$. In this case, we simply invoke Young's inequality to derive

$$\begin{aligned} \left\| \mathcal{F}^{-1} (\varphi_i \cdot f_z^{(r)}) \right\|_{L^{p_1}} &= \left\| \mathcal{F}^{-1} (\varphi_i \cdot F_i^{(r, z)}) \right\|_{L^{p_1}} \\ &\leq \left\| \mathcal{F}^{-1} \varphi_i \right\|_{L^1} \cdot \left\| \mathcal{F}^{-1} F_i^{(r, z)} \right\|_{L^{p_1}} \\ &\leq C_{\mathcal{Q}, \Phi, p_1} \cdot \left\| \mathcal{F}^{-1} F_i^{(r, z)} \right\|_{L^{p_1}} \\ &\leq C_2 C_3 C_{\mathcal{Q}, \Phi, p_1} \cdot \mu_i \end{aligned}$$

with $C_3 := 1$.

Case 2. $p_1 \in (0, 1)$. In this case, $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ is semi-structured and \mathcal{P}_{J_0} is almost subordinate to \mathcal{Q} . In fact, this is the only part of the proof where this is used. Because of this subordinateness, Lemma 2.11 implies $\text{supp } \gamma_j \subset P_j \subset Q_i^{(2k+2)*}$ for all $j \in J_i \cap J_0$, which yields $\text{supp } F_i^{(r, z)} \subset Q_i^{(2k+2)*}$. Since we clearly also have $\text{supp } \varphi_i \subset \overline{Q_i} \subset \overline{Q_i^{(2k+2)*}}$, we can

invoke Corollary 3.14 to derive

$$\begin{aligned}
 \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot f_z^{(r)} \right) \right\|_{L^{p_1}} &= \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot F_i^{(r,z)} \right) \right\|_{L^{p_1}} \\
 &\leq C_3 \cdot |\det T_i|^{\frac{1}{p_1}-1} \cdot \left\| \mathcal{F}^{-1} \varphi_i \right\|_{L^{p_1}} \cdot \left\| \mathcal{F}^{-1} F_i^{(r,z)} \right\|_{L^{p_1}} \\
 &\leq C_3 C_{\mathcal{Q}, \Phi, p_1} \cdot \left\| \mathcal{F}^{-1} F_i^{(r,z)} \right\|_{L^{p_1}} \\
 &\leq C_2 C_3 C_{\mathcal{Q}, \Phi, p_1} \cdot \mu_i
 \end{aligned}$$

for some constant $C_3 = C_3(\mathcal{Q}, k, d, p_1)$.

Taken together, both cases show that we have

$$0 \leq \varrho_i := \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot f_z^{(r)} \right) \right\|_{L^{p_1}} \leq C_4 \cdot \mu_i \quad \forall i \in I,$$

with $C_4 := C_2 C_3 C_{\mathcal{Q}, \Phi, p_1}$. By solidity of Y and because of $\mu \in Y$, this yields $\varrho = (\varrho_i)_{i \in I} \in Y$ and hence $f_z^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$ with

$$\|f_z^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^{p_1}, Y)} = \|\varrho\|_Y \leq C_4 \cdot \|\mu\|_Y = C_4 \cdot \|c\|_Y < \infty.$$

Now, we are almost done. As seen above, what we have shown implies $f_z^{(r)} \in \mathcal{D}_K^{\mathcal{Q}, p_1, Y}$ with the specific choice of z from above. Since ι is well-defined and bounded, this implies $\iota f_z^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$. Now, note for $j \in J_0$ that we have $\text{supp } \phi_j \subset P_j \subset \mathcal{O} \cap \mathcal{O}'$, so that our assumptions on ι imply for arbitrary $g \in \mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{D}_K^{\mathcal{Q}, p_1, Y}$ that

$$\langle \phi_j \cdot \iota f, g \rangle = \langle \iota f, \phi_j g \rangle = \langle f, \phi_j g \rangle = \langle \phi_j f, g \rangle,$$

i.e. $\phi_j \cdot \iota f = \phi_j f$. In particular, $\phi_j \cdot \iota f_z^{(r)} = \phi_j f_z^{(r)}$. Furthermore, $(\phi_j)_{j \in J}$ is an L^{p_2} -bounded system for \mathcal{P} , since we have $\text{supp } \phi_j \subset P_j$ for all $j \in J$ and $\sup_{j \in J} \|\mathcal{F}^{-1} \phi_j\|_{L^1} = C_2^{(1)}$, as well as

$$\sup_{j \in J} \left(|\det S_j|^{\frac{1}{q}-1} \|\mathcal{F}^{-1} \phi_j\|_{L^q} \right) = C_2^{(q)} \varepsilon^{d(1-\frac{1}{q})} \quad \text{for arbitrary } q \in (0, 1).$$

Hence, $C_5 := C_{\mathcal{P}, (\phi_i)_{i \in J}, p_2} = C(\varepsilon_{\mathcal{P}}, C_2^{(\min\{1, p_2\})}) = C(d, p_2, \varepsilon_{\mathcal{P}})$.

Next, recall from above that $f_z^{(r)} = \sum_{j \in M^{(r)}} M_{z_j}(\zeta_j \cdot \gamma_j)$, with $\text{supp } \gamma_j \subset P_j$ for all $j \in M^{(r)}$ and with $P_j^* \cap P_\ell^* = \emptyset$ for $j, \ell \in M^{(r)} \subset J^{(r)}$ with $j \neq \ell$. Finally, the “coefficient sequence” ζ is finitely supported. Thus, Lemma 6.3 (applied to \mathcal{P} instead of \mathcal{Q} , with $f = \iota f_z^{(r)}$ and $g = f_z^{(r)}$ as well as $I_0 = M^{(r)}$) yields a constant $C_6 = C_6(d, p_2, \mathcal{P}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}) > 0$ satisfying

$$\begin{aligned}
 C_5 C_6 \cdot \|\iota f_z^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{P}, L^{p_2}, Z)} &\geq \left\| (\zeta_j \cdot \|\mathcal{F}^{-1} \gamma_j\|_{L^{p_2}})_{j \in M^{(r)}} \right\|_{Z|_{M^{(r)}}} \\
 &= C_1^{(p_2)} \varepsilon^{d(1-p_2^{-1})} \cdot \left\| (\zeta_j \cdot |\det S_j|^{1-p_2^{-1}})_{j \in J_0^{(r)}} \right\|_{Z|_{J_0^{(r)}}} \\
 &= C_1^{(p_2)} \varepsilon^{d(1-p_2^{-1})} \cdot \left\| (c_j \cdot |\det S_j|^{p_1^{-1}-p_2^{-1}})_{j \in J_0^{(r)}} \right\|_{Z|_{J_0^{(r)}}}.
 \end{aligned}$$

In particular, part of the statement of Lemma 6.3 is that the sequence $(c_j \cdot |\det S_j|^{p_1^{-1}-p_2^{-1}})_{j \in J_0^{(r)}}$ is a member of $Z|_{J_0^{(r)}}$.

But using $J_0 = \bigcup_{r=1}^{r_0} J_0^{(r)}$ and the quasi-triangle inequality for Z , we get a constant $C_7 = C_7(C_Z, r_0)$ with

$$\begin{aligned} \|c\|_{(Z|_{J_0})_{|\det S_j|^{p_1^{-1}-p_2^{-1}}}} &\leq C_7 \cdot \sum_{r=1}^{r_0} \left\| \left(|\det S_j|^{p_1^{-1}-p_2^{-1}} \cdot c_j \right)_{j \in J_0^{(r)}} \right\|_{Z|_{J_0^{(r)}}} \\ &\leq \frac{C_5 C_6 C_7}{C_1^{(p_2)} \varepsilon^{d(p_2^{-1}-1)}} \cdot \sum_{r=1}^{r_0} \|f_z^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{P}, L^{p_2}, Z)} \\ &\leq \|\iota\| \frac{C_5 C_6 C_7}{C_1^{(p_2)} \varepsilon^{d(p_2^{-1}-1)}} \cdot \sum_{r=1}^{r_0} \|f_z^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^{p_1}, Y)} \\ &\leq \|\iota\| \frac{r_0 C_4 C_5 C_6 C_7}{C_1^{(p_2)} \varepsilon^{d(p_2^{-1}-1)}} \cdot \|c\|_V < \infty. \end{aligned}$$

This is precisely the desired estimate.

The final statement of the theorem is an immediate consequence of Lemmas 4.5 and 4.6. The required countability of J_0 for Lemma 4.6 holds, since \mathcal{P} is a locally finite covering of the second countable space \mathcal{O}' (cf. Lemma 2.4) with the additional property that $P_j \neq \emptyset$ for all $j \in J$, since \mathcal{P} is tight. \square

Above, we simply assumed $P_j \subset \mathcal{O} = \bigcup_{i \in I} Q_i$ for all $j \in J_0$, or—for $p_1 \in (0, 1)$ —that \mathcal{P}_{J_0} is almost subordinate to \mathcal{Q} . In our next theorem, we make the “reverse” assumption, i.e., we assume \mathcal{Q}_{I_0} to be almost subordinate to \mathcal{P} . As we will see, the techniques used in the proof are similar, but the disjointization argument becomes slightly more complex: We will apply the disjointization principle to \mathcal{P} and then use this disjointization to construct a suitable disjointization of the index set I of the covering \mathcal{Q} . We remark that the following theorem is a slightly improved version of the first part of [23, Theorem 5.3.6] from my PhD thesis.

Theorem 6.15. Let $\emptyset \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p_1, p_2 \in (0, \infty]$ and let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ be a *tight* semi-structured L^{p_1} -decomposition covering of \mathcal{O} . Furthermore, let $\mathcal{P} = (P_j)_{j \in J}$ be an L^{p_2} -decomposition covering of \mathcal{O}' . Finally, let $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ be \mathcal{Q} -regular and \mathcal{P} -regular with triangle constants $C_Y, C_Z \geq 1$, respectively.

Let $I_0 \subset I$ and assume that $\mathcal{Q}_{I_0} = (Q_i)_{i \in I_0}$ is almost subordinate to \mathcal{P} , i.e. that there is $k \in \mathbb{N}_0$ and for each $i \in I_0$ some $j_i \in J$ with $Q_i \subset P_{j_i}^{k*}$. Define

$$K := \bigcup_{i \in I_0} Q_i \subset \mathcal{O} \cap \mathcal{O}'$$

and assume that the embedding

$$\iota : \left(\mathcal{D}_K^{\mathcal{Q}, p_1, Y}, \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)} \right) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z), f \mapsto f \quad (6.37)$$

is well-defined and bounded.

Then the embedding

$$\eta : \ell_0(I_0) \cap (Y|_{I_0})_{|\det T_i|^{p_2^{-1}-p_1^{-1}}} \hookrightarrow Z \left([\ell^{p_2}(I_0 \cap I_j)]_{j \in J} \right)$$

is well-defined and bounded with

$$\|\iota\| \leq C \cdot \|\iota\|$$

for some constant

$$C = C(d, p_1, p_2, k, C_Z, \varepsilon_{\mathcal{Q}}, \mathcal{Q}, C_{\mathcal{Q}, \Phi, p_1}, \mathcal{P}, C_{\mathcal{P}, \Psi, p_2}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}).$$

Here, the L^{p_1} -BAPU $\Phi = (\varphi_i)_{i \in I}$ and the L^{p_2} -BAPU Ψ are those which are used to calculate the (quasi)-norms on the two decomposition spaces when calculating $\|\iota\|$.

Furthermore, the following hold:

- If Z satisfies the Fatou property, then η remains well-defined and bounded (with the same estimate for $\|\eta\|$), even without intersecting with $\ell_0(I_0)$.

- If $p_2 = \infty$, then η remains well-defined and bounded (with the same estimate for $\|\eta\|$) even if the “inner norm” $\ell^{p_2}(I_j \cap I_j) = \ell^\infty(I_0 \cap I_j)$ is changed to $\ell^{p_2^\vee}(I_0 \cap I_j) = \ell^1(I_0 \cap I_j)$. ◀

Remark 6.16. As for Theorem 6.13, one can obtain a generalization to the case $p_1 \neq p_2$ of the estimate $\|\delta_j\|_Z \lesssim \|\delta_i\|_Y$ for $Q_i \cap P_j$ that was shown in Lemma 6.1. Indeed, if $i_0 \in I_0$ with $\delta_{i_0} \in Y$ and $j_0 \in J$ with $Q_{i_0} \cap P_{j_0} \neq \emptyset$, then $i_0 \in I_0 \cap I_{j_0}$ and hence

$$\begin{aligned} \|\delta_{j_0}\|_Z &= \|\delta_{j_0}\|_Z \cdot \left\| (\delta_{i_0}(i))_{i \in I_0 \cap I_{j_0}} \right\|_{\ell^{p_2}} \\ &\leq \|\delta_{i_0}\|_{Z([\ell^{p_2}(I_0 \cap I_{j_0})])} \\ &\leq \|\eta\| \cdot \|\delta_{i_0}\|_{(Y|_{I_0})|_{|\det T_i|^{p_2^{-1}-p_1^{-1}}}} \\ &= \|\eta\| \cdot |\det T_{i_0}|^{p_2^{-1}-p_1^{-1}} \cdot \|\delta_{i_0}\|_Y. \end{aligned} \quad (6.38)$$

In case of $p_1 = p_2$, this again yields the estimate $\|\delta_{j_0}\|_Z \lesssim \|\delta_{i_0}\|_Y$ from Lemma 6.1. ♦

Proof of Theorem 6.15. For brevity, we write $V := (Y|_{I_0})|_{|\det T_i|^{p_2^{-1}-p_1^{-1}}} \leq \mathbb{C}^{I_0}$ and $V_0 := \ell_0(I_0) \cap V$. Now, for each $i \in I_0$, we have $Q_i \subset P_{j_i}^{k*}$ by assumption. Furthermore, $Q_i \neq \emptyset$ since \mathcal{Q} is tight. But this implies $Q_i \cap P_j \neq \emptyset$ for some $j \in j_i^{k*}$ and in particular $i \in I_0 \cap I_j$. All in all, we have shown $I_0 = \bigcup_{j \in J} (I_0 \cap I_j)$, so that $W := Z([\ell^{p_2}(I_0 \cap I_j)]_{j \in J})$ is a solid sequence space on I_0 .

Now, choose $(\gamma_i)_{i \in I}$ as in the second part of Lemma 6.12 and set $\varepsilon := \varepsilon_{\mathcal{Q}}$. Note that Lemma 6.12 is applicable, since \mathcal{Q} is tight semi-structured. By choice of $(\gamma_i)_{i \in I}$, there are certain constants $C_1 = C_1(d, p_2, \varepsilon) > 0$ and $C_2 = C_2(d, p_1, \varepsilon) > 0$ with

$$\|\mathcal{F}^{-1}\gamma_i\|_{L^{p_2}} = C_1^{-1} \cdot |\det T_i|^{1-\frac{1}{p_2}} \quad \text{and} \quad \|\mathcal{F}^{-1}\gamma_i\|_{L^{p_1}} = C_2 \cdot |\det T_i|^{1-\frac{1}{p_1}} \quad (6.39)$$

for all $i \in I$.

Let $r_0 := N_{\mathcal{P}}^{2(2k+3)+1} = N_{\mathcal{P}}^{4k+7}$. With this choice, the disjointization lemma (Lemma 2.14) yields a partition $J = \bigsqcup_{r=1}^{r_0} J^{(r)}$ such that we have $P_j^{(2k+3)*} \cap P_\ell^{(2k+3)*} = \emptyset$ for all $j, \ell \in J^{(r)}$ with $j \neq \ell$, for arbitrary $r \in r_0$.

A crucial property of the partition $J = \bigsqcup_{r=1}^{r_0} J^{(r)}$ is the following: We have

$$\forall j, \ell \in J^{(r)} \text{ with } j \neq \ell : \quad (I_j \cap I_0) \cap (I_\ell \cap I_0) = \emptyset. \quad (6.40)$$

Indeed, in case of $i \in I_j \cap I_\ell \cap I_0$, we would have $Q_i \cap P_j \neq \emptyset \neq Q_i \cap P_\ell$ —in particular, $Q_i \neq \emptyset$ —so that Lemma 2.11 would yield

$$\emptyset \neq Q_i \subset P_j^{(2k+2)*} \cap P_\ell^{(2k+2)*} \subset P_j^{(2k+3)*} \cap P_\ell^{(2k+3)*}$$

in contradiction to $j, \ell \in J^{(r)}$ with $j \neq \ell$.

These considerations show that

$$I^{(r)} := \bigcup_{j \in J^{(r)}} (I_j \cap I_0) \subset I_0$$

is well-defined. As a simple consequence of this definition, we observe

$$\forall j \in J^{(r)} : \quad I^{(r)} \cap I_j = I_0 \cap I_j. \quad (6.41)$$

Next, for $i \in I_0$ and $j \in J$ with $Q_i \cap P_j \neq \emptyset$, Lemma 2.11 yields

$$\text{supp } \gamma_i \subset Q_i \subset P_j^{(2k+2)*}$$

and Lemma 2.4 implies $\psi_j^{(2k+3)*} \equiv 1$ on $P_j^{(2k+2)*}$. Taken together, this establishes

$$\forall i \in I_0 \forall j \in J \text{ with } Q_i \cap P_j \neq \emptyset : \quad \gamma_i \psi_j^{(2k+3)*} = \gamma_i. \quad (6.42)$$

Now, let $c = (c_i)_{i \in I_0} \in \ell_0(I_0) \cap (Y|_{I_0})|_{|\det T_i|^{p_2^{-1}-p_1^{-1}}}$ be arbitrary and extend this sequence to all of I by setting $c_i := 0$ for $i \in I \setminus I_0$. Next, define $\zeta_i := |\det T_i|^{p_2^{-1}-1} \cdot |c_i|$ for $i \in I$ and set

$M := \text{supp } c = \text{supp } \zeta \subset I_0$. With these choices, define

$$\zeta^{(r)} := \zeta \cdot \mathbf{1}_{I^{(r)}} \quad \text{and} \quad g_z^{(r)} := \sum_{i \in I^{(r)}} M_{z_i} (\zeta_i \cdot \gamma_i) = \sum_{i \in I_0} M_{z_i} (\zeta_i^{(r)} \cdot \gamma_i) \quad (6.43)$$

for $r \in \underline{r}_0$ and any family $(z_i)_{i \in I_0} \in (\mathbb{R}^d)^{I_0}$ of modulations. Observe that we have $\gamma_i \in C_c^\infty(\mathcal{O})$ with $\text{supp } \gamma_i \subset Q_i \subset K$ for all $i \in I_0$. Because of $\zeta_i^{(r)} = 0$ for all but finitely many $i \in I_0$, we thus have $g_z^{(r)} \in \mathcal{D}_K^{\mathcal{Q}, p_1, Y}$, as soon as we have shown $g_z^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$.

But this is ensured by Lemma 6.2; indeed, that lemma shows $g_z^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$ and yields

$$\begin{aligned} \|g_z^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^{p_1}, Y)} &= \left\| \sum_{i \in I} M_{z_i} (\zeta_i^{(r)} \cdot \gamma_i) \right\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^{p_1}, Y)} \\ &\leq C_3 \cdot \left\| \left(\zeta_i^{(r)} \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^{p_1}} \right)_{i \in I} \right\|_Y \\ &= C_2 C_3 \cdot \left\| \left(\zeta_i^{(r)} \cdot |\det T_i|^{1-p_1^{-1}} \right)_{i \in I} \right\|_Y \\ \left(\zeta_i = |c_i| \cdot |\det T_i|^{p_2^{-1}-1} \text{ and } \zeta_i = 0 \text{ for } i \notin I_0 \right) &\leq C_2 C_3 \cdot \left\| \left(|c_i| \cdot |\det T_i|^{p_2^{-1}-1} \cdot |\det T_i|^{1-p_1^{-1}} \right)_{i \in I_0} \right\|_{Y|_{I_0}} \\ &= C_2 C_3 \cdot \|c\|_{(Y|_{I_0})_{|\det T_i|^{p_2^{-1}-p_1^{-1}}}} < \infty \end{aligned}$$

for some constant $C_3 = C_3(d, p_1, \mathcal{Q}, C_{\mathcal{Q}, \Phi, p_1}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y})$.

Since by assumption ι is well-defined and bounded, we get $g_z^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$ with

$$\|g_z^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{P}, L^{p_2}, Z)} \leq C_2 C_3 \cdot \|\iota\| \cdot \|c\|_V, \quad (6.44)$$

for every family z of modulations. Our next goal is to obtain a lower bound on $\|g_z^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{P}, L^{p_2}, Z)}$. The idea we use is similar to the proof of Lemma 6.3. But here, it is slightly easier to give a direct proof than to reduce to the setting of that lemma.

Thus, let $j \in J^{(r)}$ and $i \in I^{(r)} \subset I_0$ be arbitrary. There are two cases:

Case 1. We have $i \in I_j$. In this case, equation (6.42) implies

$$\begin{aligned} \psi_j^{(2k+3)*} \cdot M_{z_i} (\zeta_i \cdot \gamma_i) &= M_{z_i} (\zeta_i \cdot \psi_j^{(2k+3)*} \cdot \gamma_i) \\ &= M_{z_i} (\zeta_i \cdot \gamma_i). \end{aligned} \quad (6.45)$$

Case 2. We have $i \notin I_j$. Because of $i \in I^{(r)} = \biguplus_{\ell \in J^{(r)}} (I_0 \cap I_\ell)$, this implies $i \in I_\ell$ for some $\ell \in J^{(r)} \setminus \{j\}$. But since $j, \ell \in J^{(r)}$ with $j \neq \ell$, we get $P_j^{(2k+3)*} \cap P_\ell^{(2k+3)*} = \emptyset$ which entails $\psi_j^{(2k+3)*} \psi_\ell^{(2k+3)*} \equiv 0$ and thus (using equation (6.45) for ℓ instead of j) that

$$\psi_j^{(2k+3)*} \cdot M_{z_i} (\zeta_i \cdot \gamma_i) = \psi_j^{(2k+3)*} \cdot \psi_\ell^{(2k+3)*} \cdot M_{z_i} (\zeta_i \cdot \gamma_i) \equiv 0.$$

In summary, these considerations imply

$$\psi_j^{(2k+3)*} \cdot M_{z_i} (\zeta_i \cdot \gamma_i) = \begin{cases} M_{z_i} (\zeta_i \cdot \gamma_i), & \text{if } i \in I_j, \\ 0, & \text{if } i \notin I_j \end{cases} \quad (6.46)$$

for arbitrary $j \in J^{(r)}$, $i \in I^{(r)}$ and $r \in \underline{r}_0$.

Next, a combination of Theorem 3.17 and Remark 3.16 yields a constant

$$C_4 = C_4(\mathcal{P}, p_2, k, d, C_{\mathcal{P}, \Psi, p_2}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}) > 0$$

satisfying

$$\left\| \left(\|\mathcal{F}^{-1} (\psi_j^{(2k+3)*} \cdot g)\|_{L^{p_2}} \right)_{j \in J} \right\|_Z \leq C_4 \cdot \|g\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{P}, L^{p_2}, Z)}$$

for all $g \in \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$. In particular, we have $(\|\mathcal{F}^{-1}(\psi_j^{(2k+3)*} \cdot g)\|_{L^{p_2}})_{j \in J} \in Z$ for each such g . By applying this to $g = g_z^{(r)}$, we derive

$$\begin{aligned}
 \|g_z^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{P}, L^{p_2}, Z)} &\geq C_4^{-1} \cdot \left\| \left(\|\mathcal{F}^{-1}(\psi_j^{(2k+3)*} \cdot g_z^{(r)})\|_{L^{p_2}} \right)_{j \in J} \right\|_Z \\
 &\geq C_4^{-1} \cdot \left\| \left(\|\mathcal{F}^{-1}(\psi_j^{(2k+3)*} \cdot g_z^{(r)})\|_{L^{p_2}} \right)_{j \in J^{(r)}} \right\|_{Z|_{J^{(r)}}} \\
 &= C_4^{-1} \cdot \left\| \left(\left\| \mathcal{F}^{-1} \left[\sum_{i \in I^{(r)}} \psi_j^{(2k+3)*} \cdot M_{z_i}(\zeta_i \cdot \gamma_i) \right] \right\|_{L^{p_2}} \right)_{j \in J^{(r)}} \right\|_{Z|_{J^{(r)}}} \\
 (\text{eq. (6.46)}) &= C_4^{-1} \cdot \left\| \left(\left\| \mathcal{F}^{-1} \left[\sum_{i \in I^{(r)} \cap I_j} M_{z_i}(\zeta_i \cdot \gamma_i) \right] \right\|_{L^{p_2}} \right)_{j \in J^{(r)}} \right\|_{Z|_{J^{(r)}}}. \tag{6.47}
 \end{aligned}$$

At this point, we need to separate two cases, namely $p_2 = \infty$ and $p_2 < \infty$.

We begin with the case $p_2 = \infty$, in which we will even show the (stronger) estimate with $\ell^{p_2}(I_0 \cap I_j)$ replaced by $\ell^1(I_0 \cap I_j)$, cf. the last part of the theorem. Recall that we obtained the γ_i from Lemma 6.12. In particular, $\gamma_i \geq 0$ for all $i \in I$, which yields

$$(\mathcal{F}^{-1}\gamma_i)(0) = \int_{\mathbb{R}^d} \gamma_i(\xi) \, d\xi = \|\gamma_i\|_{L^1} \geq \|\mathcal{F}^{-1}\gamma_i\|_{L^\infty} = \|\mathcal{F}^{-1}\gamma_i\|_{L^{p_2}} = C_1^{-1} \cdot |\det T_i|^{1-p_2^{-1}},$$

as a consequence of the Riemann-Lebesgue lemma and of equation (6.39). Thus, if we simply set $z_i = 0$ for all $i \in I$, we get for each $j \in J^{(r)}$ that

$$\begin{aligned}
 \left\| \mathcal{F}^{-1} \left[\sum_{i \in I^{(r)} \cap I_j} M_{z_i}(\zeta_i \cdot \gamma_i) \right] \right\|_{L^{p_2}} &= \left\| \sum_{i \in I^{(r)} \cap I_j} \zeta_i \cdot \mathcal{F}^{-1}\gamma_i \right\|_{L^\infty} \\
 (\text{function is continuous}) &\geq \left| \sum_{i \in I^{(r)} \cap I_j} \zeta_i \cdot (\mathcal{F}^{-1}\gamma_i)(0) \right| \\
 (\zeta_i \geq 0) &\geq C_1^{-1} \cdot \sum_{i \in I^{(r)} \cap I_j} \zeta_i |\det T_i|^{1-p_2^{-1}} \\
 &= C_1^{-1} \cdot \sum_{i \in I^{(r)} \cap I_j} \left[|c_i| \cdot |\det T_i|^{p_2^{-1}-1} \cdot |\det T_i|^{1-p_2^{-1}} \right] \\
 &= C_1^{-1} \cdot \|c\|_{\ell^1(I_j \cap I^{(r)})} \\
 (\text{eq. (6.41) and } j \in J^{(r)}) &= C_1^{-1} \cdot \|c\|_{\ell^1(I_j \cap I_0)}.
 \end{aligned}$$

By solidity of Z and in view of equation (6.47), we have thus shown $(\|c\|_{\ell^1(I_j \cap I_0)} \cdot \mathbf{1}_{J^{(r)}}(j))_{j \in J} \in Z$ with

$$\|g_z^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{P}, L^{p_2}, Z)} \geq \frac{1}{C_1 C_4} \cdot \left\| \left(\|c\|_{\ell^1(I_j \cap I_0)} \right)_{j \in J^{(r)}} \right\|_{Z|_{J^{(r)}}}.$$

Now, using the quasi-triangle inequality for Z and $J = \biguplus_{r=1}^{r_0} J^{(r)}$, we finally get $c \in Z \left([\ell^1(I_j \cap I_0)]_{j \in J} \right)$, with

$$\begin{aligned} \|c\|_{Z([\ell^1(I_j \cap I_0)]_{j \in J})} &= \left\| \left(\|c\|_{\ell^1(I_j \cap I_0)} \right)_{j \in J} \right\|_Z \\ &\leq C_5 \cdot \sum_{r=1}^{r_0} \left\| \left(\|c\|_{\ell^1(I_j \cap I_0)} \right)_{j \in J^{(r)}} \right\|_{Z|_{J^{(r)}}} \\ &\leq C_1 C_4 C_5 \cdot \sum_{r=1}^{r_0} \|g_z^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{P}, L^{p_2}, Z)} \\ (\text{eq. (6.44)}) &\leq C_1 C_2 C_3 C_4 C_5 r_0 \cdot \|\iota\| \cdot \|c\|_V < \infty \end{aligned}$$

for some constant $C_5 = C_5(r_0, C_Z) = C_5(k, N_{\mathcal{P}}, C_Z)$. This establishes boundedness of η (with the “inner norm” $\ell^{p_2}(I_0 \cap I_j)$ replaced by $\ell^1(I_0 \cap I_j)$), as desired. In particular, this establishes the last part of the theorem.

Now, let us consider the general case. The proof given here is also applicable for $p_2 = \infty$, but yields a weaker conclusion (using $\ell^{p_2}(I_0 \cap I_j)$ instead of $\ell^1(I_0 \cap I_j)$). Here, we cannot simply choose all modulations $z_i = 0$. Instead, we apply Corollary 6.6 to the family $(h_i)_{i \in M}$ defined by $h_i := \zeta_i^{(r)} \cdot \gamma_i$ for $i \in M = \text{supp } \zeta$. This yields a family $z^{(r)} = (z_i^{(r)})_{i \in M}$ of modulations so that

$$\begin{aligned} \left\| \mathcal{F}^{-1} \left(\sum_{i \in S} M_{z_i^{(r)}} \left(\zeta_i^{(r)} \cdot \gamma_i \right) \right) \right\|_{L^{p_2}} &\geq \frac{1}{2} \cdot \left\| \left(\left\| \mathcal{F}^{-1} \left(\zeta_i^{(r)} \cdot \gamma_i \right) \right\|_{L^{p_2}} \right)_{i \in S} \right\|_{\ell^{p_2}} \\ (\text{eq. (6.39) and } \zeta_i^{(r)} = 0 \text{ for } i \notin I^{(r)}) &= \frac{1}{2C_1} \cdot \left\| \left(|c_i| \cdot |\det T_i|^{p_2^{-1}-1} \cdot |\det T_i|^{1-\frac{1}{p_2}} \right)_{i \in S \cap I^{(r)}} \right\|_{\ell^{p_2}} \\ &= \frac{1}{2C_1} \cdot \|(c_i)_{i \in S \cap I^{(r)}}\|_{\ell^{p_2}} \end{aligned} \quad (6.48)$$

holds for all subsets $S \subset M$. For simplicity, set $z_i^{(r)} := 0$ for $i \in I \setminus M$.

With this choice of $z^{(r)}$, we have

$$\begin{aligned} \left\| \mathcal{F}^{-1} \left[\sum_{i \in I^{(r)} \cap I_j} M_{z_i^{(r)}} (\zeta_i \cdot \gamma_i) \right] \right\|_{L^{p_2}} &= \left\| \mathcal{F}^{-1} \left[\sum_{i \in M \cap I_j} M_{z_i^{(r)}} (\zeta_i^{(r)} \cdot \gamma_i) \right] \right\|_{L^{p_2}} \\ &\geq \frac{1}{2C_1} \cdot \|(c_i)_{i \in M \cap I_j \cap I^{(r)}}\|_{\ell^{p_2}} \\ (c_i = 0 \text{ for } i \in I \setminus M) &= \frac{1}{2C_1} \cdot \|(c_i)_{i \in I_j \cap I^{(r)}}\|_{\ell^{p_2}} \\ (\text{eq. (6.41)}) &= \frac{1}{2C_1} \cdot \|(c_i)_{i \in I_j \cap I_0}\|_{\ell^{p_2}} \end{aligned}$$

for each $j \in J^{(r)}$. By solidity of Z and using eq. (6.47), we thus get $\left(\|c\|_{\ell^{p_2}(I_j \cap I_0)} \cdot \mathbf{1}_{J^{(r)}}(j) \right)_{j \in J} \in Z$ with

$$\|g_z^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{P}, L^{p_2}, Z)} \geq \frac{1}{2C_1 C_4} \cdot \left\| \left(\|c\|_{\ell^{p_2}(I_j \cap I_0)} \right)_{j \in J^{(r)}} \right\|_{Z|_{J^{(r)}}}.$$

Now, using the quasi-triangle inequality for Z and $J = \biguplus_{r=1}^{r_0} J^{(r)}$, we finally get $c \in Z \left([\ell^{p_2}(I_j \cap I_0)]_{j \in J} \right)$, with

$$\begin{aligned} \|c\|_{Z([\ell^{p_2}(I_j \cap I_0)]_{j \in J})} &= \left\| \left(\|c\|_{\ell^{p_2}(I_j \cap I_0)} \right)_{j \in J} \right\|_Z \\ &\leq C_5 \cdot \sum_{r=1}^{r_0} \left\| \left(\|c\|_{\ell^{p_2}(I_j \cap I_0)} \right)_{j \in J^{(r)}} \right\|_{Z|_{J^{(r)}}} \\ &\leq 2C_1 C_4 C_5 \cdot \sum_{r=1}^{r_0} \left\| g_{z^{(r)}}^{(r)} \right\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{P}, L^{p_2}, Z)} \\ (\text{eq. (6.44)}) &\leq 2C_1 C_2 C_3 C_4 C_5 r_0 \cdot \|\iota\| \cdot \|c\|_V < \infty \end{aligned}$$

for the same constant $C_5 = C_5(k, N_{\mathcal{P}}, C_Z)$ as for $p_2 = \infty$. This completes the proof of the main statement of the theorem.

Finally, if Z satisfies the Fatou property, it is an immediate consequence of Lemmas 4.5 and 4.6 that

$$\tilde{\eta} : (Y|_{I_0})_{|\det T_1|^{p_2^{-1}-p_1^{-1}}} \hookrightarrow Z \left([\ell^{p_2}(I_j \cap I_0)]_{j \in J} \right)$$

is bounded if and only if η is, with $\|\tilde{\eta}\| = \|\eta\|$. The required countability of I_0 for Lemma 4.6 holds, since \mathcal{Q} is a locally finite covering of the second countable space \mathcal{O} (cf. Lemma 2.4) with the additional property that $Q_i \neq \emptyset$ for all $i \in I$, since \mathcal{Q} is tight. \square

6.4. Further necessary conditions in case of $p_1 = p_2$. In the previous subsections, we obtained necessary conditions for existence (of slight generalizations) of the embedding

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z), \quad (6.49)$$

either assuming that (a subfamily of) $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ is almost subordinate to $\mathcal{P} = (S_j P'_j + c_j)_{j \in J}$, or vice versa. Slightly simplified, the derived necessary conditions were $p_1 \leq p_2$ and

$$Y_{|\det T_i|^{p_2^{-1}-p_1^{-1}}} \hookrightarrow Z \left([\ell^{p_2}(I_j)]_{j \in J} \right) \quad (6.50)$$

in case \mathcal{Q} is almost subordinate to \mathcal{P} and

$$Y([\ell^{p_1}(J_i)]_{i \in I}) \hookrightarrow Z_{|\det S_j|^{p_1^{-1}-p_2^{-1}}} \quad (6.51)$$

in case \mathcal{P} is almost subordinate to \mathcal{Q} . Indeed, at least if Z satisfies the Fatou property and if \mathcal{Q}, \mathcal{P} are tight semi-structured, these embeddings are consequences of Theorem 6.15 (with $I_0 = I$) or of Theorem 6.13 (with $J_0 = J$), respectively, while $p_1 \leq p_2$ is a consequence of Lemma 6.1.

Conversely, if \mathcal{Q} is almost subordinate to \mathcal{P} , then Corollary 5.7 shows that a *sufficient* condition for existence of the embedding (6.49) is $p_1 \leq p_2$ and

$$Y_{|\det T_i|^{p_2^{-1}-p_1^{-1}}} \hookrightarrow Z \left([\ell^{p_2^\vee}(I_j)]_{j \in J} \right).$$

If instead \mathcal{P} is almost subordinate to \mathcal{Q} , then Corollary 5.11 shows—at least for $p_1 \geq 1$ —that $p_1 \leq p_2$ and

$$Y([\ell^{p_1^\Delta}(J_i)]_{i \in I}) \hookrightarrow Z_{|\det S_j|^{p_1^{-1}-p_2^{-1}}}$$

are sufficient for existence of the embedding (6.49).

In summary, we thus achieve a *complete characterization* of the existence of the embedding (6.49) in terms of embeddings for discrete (nested) sequence spaces in the following cases:

- If \mathcal{Q} is almost subordinate to \mathcal{P} and if $p_2 \in (0, 2] \cup \{\infty\}$. Indeed, for $p_2 \in (0, 2]$, we have $p_2^\vee = p_2$, so that both conditions coincide. Finally, in case of $p_2 = \infty$, Theorem 6.15 shows that we can replace $\ell^{p_2}(I_j)$ by $\ell^1(I_j) = \ell^{\infty^\vee}(I_j)$ in the necessary condition (6.50).
- If \mathcal{P} is almost subordinate to \mathcal{Q} and if $p_1 \in [2, \infty]$, since this implies $p_1 \geq 1$, so that the sufficient condition from above is applicable and since we have $p_1^\Delta = p_1$ in this range.

In the remaining cases, there is a gap between the necessary and sufficient conditions. Under additional hypotheses, this gap can be closed, as the next subsection shows.

In this subsection, our aim is slightly different: Note that we always have $p_1^\Delta \geq 2$ and $p_1^\nabla \leq 2$. Thus, there seems to be something special about the “critical” exponent 2. In this subsection, we will show that this is indeed the case. Assuming $p_1 = p_2$, we will show that

$$Y \hookrightarrow Z \left([\ell^2(I_j)]_{j \in J} \right) \quad \text{or} \quad Y \left([\ell^2(J_i)]_{i \in I} \right) \hookrightarrow Z$$

are always necessary conditions for existence of the embedding (6.49), respectively, if \mathcal{Q} is almost subordinate to \mathcal{P} or vice versa. This strengthens the necessary conditions (6.50) and (6.51), while still not reaching the “full power” of the sufficient conditions. Thus, we *tighten* the gap, but fail to close it completely. As remarked above, closing the gap completely—but only under additional assumptions—is the goal of the next subsection.

We begin with the following theorem which is analogous to Theorem 6.13 from the previous subsection. A curious feature of the present result is that we do not need to assume \mathcal{Q} or \mathcal{P} to be tight—not even semi-structured. Furthermore, we do not need to assume \mathcal{P} to be almost subordinate to \mathcal{Q} , not even for $p \in (0, 1)$, in contrast to Theorem 6.13.

Theorem 6.17. Let $\emptyset \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p \in (0, \infty]$ and let $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ be two L^p -decomposition coverings of \mathcal{O} and \mathcal{O}' , respectively. Finally, let $Y \leq \mathbb{C}^I$ be \mathcal{Q} -regular and $Z \leq \mathbb{C}^J$ be \mathcal{P} -regular.

Let $J_0 \subset J$ and assume that P_j is open and $P_j \subset \mathcal{O}$ for each $j \in J_0$. Define $K := \bigcup_{j \in J_0} P_j \subset \mathcal{O} \cap \mathcal{O}'$. If the map

$$\iota : \left(\mathcal{D}_K^{\mathcal{Q}, p, Y}, \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} \right) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z), f \mapsto f \quad (6.52)$$

is well-defined and continuous, then so is the embedding

$$\eta : \ell_0(J_0) \cap Y \left([\ell^{\max\{2, p\}}(J_0 \cap J_i)]_{i \in I} \right) \hookrightarrow Z,$$

with $\|\eta\| \leq C \cdot \|\iota\|$ for some constant

$$C = C(d, p, C_Z, \mathcal{Q}, \mathcal{P}, C_{\mathcal{Q}, \Phi, p}, C_{\mathcal{P}, \Psi, p}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}) > 0.$$

Here, the L^p -BAPUs $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_j)_{j \in J}$ are those which are used to compute the (quasi)-norms on the respective decomposition spaces when computing the norm $\|\iota\|$.

Finally, if Z satisfies the Fatou property, the same statement as above also holds for the embedding $\eta : Y \left([\ell^{\max\{2, p\}}(J_0 \cap J_i)]_{i \in I} \right) \hookrightarrow Z$, i.e. without restricting to $\ell_0(J_0)$. \blacktriangleleft

Remark. Note that we always have norm-decreasing embeddings $\ell^2(J_0 \cap J_i) \hookrightarrow \ell^{\max\{2, p\}}(J_0 \cap J_i)$ and $\ell^p(J_0 \cap J_i) \hookrightarrow \ell^{\max\{2, p\}}(J_0 \cap J_i)$. Thus, it is easy to see that the theorem implies in particular that

$$\begin{aligned} \eta_1 : \ell_0(J_0) \cap Y \left([\ell^p(J_0 \cap J_i)]_{i \in I} \right) &\hookrightarrow Z, \\ \eta_2 : \ell_0(J_0) \cap Y \left([\ell^2(J_0 \cap J_i)]_{i \in I} \right) &\hookrightarrow Z \end{aligned}$$

are well-defined and bounded with $\|\eta_\ell\| \leq \|\eta\|$ for $\ell \in \{1, 2\}$. \blacklozenge

Proof. For the sake of brevity, set $s := \max\{2, p\} \in [2, \infty]$ and $V := Y \left([\ell^s(J_0 \cap J_i)]_{i \in I} \right)$, as well as $V_0 := \ell_0(J_0) \cap V$. Furthermore, fix a nonzero function $\gamma \in C_c^\infty(B_1(0))$ for the rest of the proof. Finally, let $r_0 := N_{\mathcal{P}}^3$. With this choice, Lemma 2.14 yields a partition $J = \bigsqcup_{r=1}^{r_0} J_{(r)}$ such that $P_j^* \cap P_\ell^* = \emptyset$ holds for all $j, \ell \in J_{(r)}$ with $j \neq \ell$ and arbitrary $r \in \underline{r_0}$.

Note that we have $\emptyset \neq P_j \subset \mathcal{O}$ for all $j \in J_0$ (by definition of an admissible covering, all sets of \mathcal{P} are nonempty), so that we get $Q_i \cap P_j \neq \emptyset$ for some $i \in I$, since \mathcal{Q} covers I . Hence, $J_0 = \bigcup_{i \in I} (J_0 \cap J_i)$, so that V is a solid sequence space over J_0 .

Let $c = (c_j)_{j \in J_0} \in V_0$ be arbitrary and let $M := \text{supp } c$, which is a finite subset of J_0 . Since $M \subset J_0$ is finite and since each set P_j with $j \in J_0$ is open, there is some $\delta = \delta(c) > 0$ and for each $j \in M$ some $\xi_j \in \mathbb{R}^d$ (possibly depending on c) with $B_\delta(\xi_j) \subset P_j \subset \mathcal{O}$. Now, Lemma 2.4 shows that $\mathcal{Q}^\circ = (Q_i^\circ)_{i \in I}$ covers \mathcal{O} , so that for each $j \in M$ there is some $\ell_j \in I$ with $\xi_j \in Q_{\ell_j}^\circ$. This yields some $\varepsilon_j \in (0, \delta)$

satisfying $B_{\varepsilon_j}(\xi_j) \subset Q_{\ell_j}$. Again by finiteness of M , we see that $\varepsilon := \min_{j \in M} \varepsilon_j \in (0, \delta)$ is well-defined with $B_{\varepsilon}(\xi_j) \subset P_j \cap Q_{\ell_j}$ for all $j \in M$.

It is important to observe that the quantities $(\xi_j)_{j \in M}$ and $\varepsilon > 0$ depend highly on the sequence $c = (c_j)_{j \in J_0}$. Nevertheless, all constants C_1, C_2, \dots chosen in the remainder of the proof will be independent of the sequence c . Finally, we will see that the precise choice of ξ_j will be immaterial for the resulting estimates and that all occurrences of ε will cancel in the end.

For $j \in M$, let $\gamma_j := L_{\xi_j}(\gamma(\varepsilon^{-1} \cdot))$ and note $\text{supp } \gamma_j \subset B_{\varepsilon}(\xi_j) \subset P_j \cap Q_{\ell_j}$, as well as

$$\mathcal{F}^{-1} \gamma_j = \varepsilon^d \cdot M_{\xi_j}[(\mathcal{F}^{-1} \gamma)(\varepsilon \cdot)], \quad (6.53)$$

which implies

$$\|\mathcal{F}^{-1} \gamma_j\|_{L^p} = C_1 \cdot \varepsilon^{d(1-\frac{1}{p})} \quad \text{for} \quad C_1 := \|\mathcal{F}^{-1} \gamma\|_{L^p}. \quad (6.54)$$

Note $C_1 = C_1(d, p)$.

Now, we make some technical observations which will become important in the remainder of the proof. First, for $\ell \in I$, we define

$$M^{(\ell)} := \{j \in M \mid \ell_j = \ell\}$$

and note $M = \biguplus_{\ell \in I} M^{(\ell)}$, as well as $j \in M^{(\ell_j)}$ for all $j \in M$. Next, suppose that we have

$$f = \sum_{j \in M} M_{z_j}(d_j \cdot \gamma_j)$$

for some sequence $(d_j)_{j \in M} \in \mathbb{C}^M$ and $(z_j)_{j \in M} \in (\mathbb{R}^d)^M$. We are interested in a simplified form of $\varphi_i \cdot f$, for arbitrary $i \in I$.

To this end, note for $j \in M$ that we have $\varphi_i \gamma_j \equiv 0$ unless $\emptyset \neq Q_i \cap \text{supp } \gamma_j \subset Q_i \cap Q_{\ell_j}$. But in this case, we get $\ell_j \in i^*$ and thus $j \in M^{(\ell_j)} \subset \bigcup_{\ell \in i^*} M^{(\ell)}$. Thus, since the sequence $(M^{(\ell)})_{\ell \in I}$ is pairwise disjoint, we finally get

$$\begin{aligned} \varphi_i \cdot f &= \varphi_i \cdot \sum_{j \in M} M_{z_j}(d_j \cdot \gamma_j) = \sum_{j \in M} d_j \cdot M_{z_j}(\varphi_i \cdot \gamma_j) \\ &= \sum_{j \in \bigcup_{\ell \in i^*} M^{(\ell)}} d_j \cdot M_{z_j}(\varphi_i \cdot \gamma_j) \\ &= \sum_{\ell \in i^*} \sum_{j \in M^{(\ell)}} d_j \cdot M_{z_j}(\varphi_i \cdot \gamma_j) \\ &\left(\text{with } f^{(\ell)} := \sum_{j \in M^{(\ell)}} M_{z_j}(d_j \cdot \gamma_j) \right) = \sum_{\ell \in i^*} [\varphi_i \cdot f^{(\ell)}]. \end{aligned} \quad (6.55)$$

Now, we want to derive an estimate for $\|\mathcal{F}^{-1}(\varphi_i \cdot f)\|_{L^p}$. In case of $p \in [1, \infty]$, this is straightforward: Using the triangle inequality for L^p and Young's inequality $L^1 * L^p \hookrightarrow L^p$, we get

$$\begin{aligned} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \sum_{j \in M} M_{z_j}(d_j \cdot \gamma_j) \right) \right\|_{L^p} &\leq \sum_{\ell \in i^*} \left\| \mathcal{F}^{-1} \left[\varphi_i \cdot \sum_{j \in M^{(\ell)}} M_{z_j}(d_j \cdot \gamma_j) \right] \right\|_{L^p} \\ &\leq \sum_{\ell \in i^*} \|\mathcal{F}^{-1} \varphi_i\|_{L^p} \cdot \left\| \mathcal{F}^{-1} \left[\sum_{j \in M^{(\ell)}} M_{z_j}(d_j \cdot \gamma_j) \right] \right\|_{L^p} \\ &\leq C_{\mathcal{Q}, \Phi, p} \cdot \sum_{\ell \in i^*} \left\| \mathcal{F}^{-1} \left[\sum_{j \in M^{(\ell)}} M_{z_j}(d_j \cdot \gamma_j) \right] \right\|_{L^p}. \end{aligned}$$

In case of $p \in (0, 1)$, the argument is slightly more involved: For $\ell \in i^*$ and $j \in M^{(\ell)}$, we have $\ell_j = \ell \in i^*$ and hence

$$\text{supp } \gamma_j \subset P_j \cap Q_{\ell_j} \subset Q_{\ell} \subset Q_i^* \subset \overline{Q_i^*},$$

which yields $\text{supp } f^{(\ell)} \subset \overline{Q_i^*}$, since $j \in M^{(\ell)}$ was arbitrary. Furthermore, $\text{supp } \varphi_i \subset \overline{Q_i} \subset \overline{Q_i^*}$. Finally, because \mathcal{Q} is an L^p -decomposition covering of \mathcal{O} and since $p \in (0, 1)$, $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ is semi-structured. All in all, we can apply Corollary 3.14, which yields a constant $C_2 = C_2(\mathcal{Q}, d, p)$ satisfying

$$\|\mathcal{F}^{-1}(\varphi_i \cdot f^{(\ell)})\|_{L^p} \leq C_2 \cdot |\det T_i|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}\varphi_i\|_{L^p} \cdot \|\mathcal{F}^{-1}f^{(\ell)}\|_{L^p} \leq C_2 C_{\mathcal{Q}, \Phi, p} \cdot \|\mathcal{F}^{-1}f^{(\ell)}\|_{L^p}.$$

Finally, using the uniform estimate $|i^*| \leq N_{\mathcal{Q}}$ and the quasi-triangle inequality for L^p , we obtain a constant $C_3 = C_3(N_{\mathcal{Q}}, p)$ with

$$\begin{aligned} \left\| \mathcal{F}^{-1} \left(\varphi_i \cdot \sum_{j \in M} M_{z_j} (d_j \cdot \gamma_j) \right) \right\|_{L^p} &\stackrel{\text{eq. (6.55)}}{=} \left\| \mathcal{F}^{-1} \left(\sum_{\ell \in i^*} \varphi_i \cdot f^{(\ell)} \right) \right\|_{L^p} \\ &\leq C_3 \cdot \sum_{\ell \in i^*} \|\mathcal{F}^{-1}(\varphi_i \cdot f^{(\ell)})\|_{L^p} \\ &\leq C_2 C_3 C_{\mathcal{Q}, \Phi, p} \cdot \sum_{\ell \in i^*} \|\mathcal{F}^{-1}f^{(\ell)}\|_{L^p} \\ &= C_2 C_3 C_{\mathcal{Q}, \Phi, p} \cdot \sum_{\ell \in i^*} \left\| \mathcal{F}^{-1} \left[\sum_{j \in M^{(\ell)}} M_{z_j} (d_j \cdot \gamma_j) \right] \right\|_{L^p}. \end{aligned} \quad (6.56)$$

Thus, if we set $C_2 := C_3 := 1$ for $p \in [1, \infty]$, we see that the previous estimate remains true for all $p \in (0, \infty]$.

Now, we divide the proof into two parts. For the first part, we assume $p \leq 2 < \infty$, so that we get $s = \max\{2, p\} = 2$. In the second part, we will consider the case $p > 2$, where $s = p$.

Part 1: Here, we have $p \leq 2 < \infty$. Let us fix some $r \in r_0$. Now, for arbitrary $\ell \in I$, we consider the random variable $\omega^{(\ell)} = \omega^{(\ell, r)} = (\omega_j^{(\ell)})_{j \in M^{(\ell)} \cap J_{(r)}} \in \{\pm 1\}^{M^{(\ell)} \cap J_{(r)}}$, which we take to be uniformly distributed in $\{\pm 1\}^{M^{(\ell)} \cap J_{(r)}}$. The expectation in the following computation is to be understood with respect to $\omega^{(\ell)}$. Let $C_4 = C_4(p)$ be the constant supplied by Khintchine's inequality (Theorem 6.7). We then have (since we can interchange the expectation with the integral by Fubini's theorem—or since it is just a finite sum)

$$\begin{aligned} \mathbb{E} \left\| \mathcal{F}^{-1} \left[\sum_{j \in M^{(\ell)} \cap J_{(r)}} c_j \omega_j^{(\ell)} \cdot \gamma_j \right] \right\|_{L^p}^p &= \mathbb{E} \int_{\mathbb{R}^d} \left| \sum_{j \in M^{(\ell)} \cap J_{(r)}} c_j \omega_j^{(\ell)} \cdot (\mathcal{F}^{-1}\gamma_j)(x) \right|^p dx \\ (\text{eq. (6.53)}) &= \varepsilon^{dp} \cdot \int_{\mathbb{R}^d} \mathbb{E} \left| \sum_{j \in M^{(\ell)} \cap J_{(r)}} \omega_j^{(\ell)} \cdot c_j \cdot e^{2\pi i \langle \xi_j, x \rangle} \cdot (\mathcal{F}^{-1}\gamma)(\varepsilon x) \right|^p dx \\ (\text{by Khintchine's ineq.}) &\leq C_4 \varepsilon^{dp} \cdot \int_{\mathbb{R}^d} \left(\sum_{j \in M^{(\ell)} \cap J_{(r)}} |c_j \cdot e^{2\pi i \langle \xi_j, x \rangle} \cdot (\mathcal{F}^{-1}\gamma)(\varepsilon x)|^2 \right)^{p/2} dx \\ &= C_4 \varepsilon^{dp} \cdot \left[\sum_{j \in M^{(\ell)} \cap J_{(r)}} |c_j|^2 \right]^{p/2} \cdot \int_{\mathbb{R}^d} |(\mathcal{F}^{-1}\gamma)(\varepsilon x)|^p dx \\ (\text{since } C_1 = \|\mathcal{F}^{-1}\gamma\|_{L^p}) &= C_1^p C_4 \cdot \varepsilon^{d(p-1)} \cdot \left\| (c_j)_{j \in M^{(\ell)} \cap J_{(r)}} \right\|_{\ell^2}^p \\ &\leq \left[C_1 C_4^{1/p} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \left\| (c_j)_{j \in J_0 \cap J_\ell} \right\|_{\ell^2} \right]^p. \end{aligned} \quad (6.57)$$

Here, the last step used that we have $\xi_j \in Q_{\ell_j} \cap P_j = Q_\ell \cap P_j$ and hence $j \in M \cap J_\ell \subset J_0 \cap J_\ell$ for $j \in M^{(\ell)}$.

Estimate (6.57) yields a (deterministic) realization $\theta^{(\ell)} = \theta^{(\ell,r)} = (\theta_j^{(\ell)})_{j \in M^{(\ell)} \cap J_{(r)}} \in \{\pm 1\}^{M^{(\ell)} \cap J_{(r)}}$ with

$$\left\| \mathcal{F}^{-1} \left[\sum_{j \in M^{(\ell)} \cap J_{(r)}} c_j \theta_j^{(\ell)} \cdot \gamma_j \right] \right\|_{L^p} \leq C_1 C_4^{1/p} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \left\| (c_j)_{j \in J_0 \cap J_\ell} \right\|_{\ell^2}. \quad (6.58)$$

Since we can choose such a realization for every $\ell \in I$ and since the sets $(M^{(\ell)})_{\ell \in I}$ are pairwise disjoint with $M = \biguplus_{\ell \in I} M^{(\ell)}$, we obtain a well-defined “global” realization $\theta = \theta^{(r)} \in \{\pm 1\}^{M \cap J_{(r)}}$ defined by $\theta_j := \theta_j^{(\ell_j)}$ for every $j \in M \cap J_{(r)}$. Note that we have $\theta_j = \theta_j^{(\ell)} = \theta_j^{(\ell,r)}$ for every $j \in M^{(\ell)} \cap J_{(r)}$.

Now, define

$$g^{(r)} = g_\theta^{(r)} := \sum_{j \in M \cap J_{(r)}} c_j \theta_j \cdot \gamma_j. \quad (6.59)$$

Observe $\text{supp } \gamma_j \subset P_j \subset K$ for each $j \in M \subset J_0$ and hence $\text{supp } g^{(r)} \subset K$, since the sum is finite. Thus, $g^{(r)} \in \mathcal{D}_K^{\mathcal{Q},p,Y}$, as soon as we have shown $g^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$.

To see $g^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$, define $\varrho = (\varrho_\ell)_{\ell \in I}$ by $\varrho_\ell := \left\| (c_j)_{j \in J_0 \cap J_\ell} \right\|_{\ell^2}$. Note that we have $\varrho \in Y$ with $\|\varrho\|_Y = \|c\|_V$, since $c \in V = Y([\ell^s(J_0 \cap J_i)]_{i \in I})$ and because of $s = 2$, since $p \leq 2$. Since Y is \mathcal{Q} -regular, this also implies $\Gamma_{\mathcal{Q}}\varrho \in Y$, which will soon become helpful.

Indeed, let $i \in I$. Using estimate (6.56) (with $z_j = 0$ for all $j \in M$ and $d_j = c_j \theta_j \cdot \mathbf{1}_{M \cap J_{(r)}}(j)$), we derive

$$\begin{aligned} \left\| \mathcal{F}^{-1}(\varphi_i \cdot g^{(r)}) \right\|_{L^p} &\leq C_2 C_3 C_{\mathcal{Q},\Phi,p} \cdot \sum_{\ell \in i^*} \left\| \mathcal{F}^{-1} \left[\sum_{j \in M^{(\ell)} \cap J_{(r)}} c_j \theta_j \cdot \gamma_j \right] \right\|_{L^p} \\ (\text{eq. (6.58) and } \theta_j = \theta_j^{(\ell)} \text{ for } j \in M^{(\ell)} \cap J_{(r)}) &\leq C_1 C_2 C_3 C_4^{1/p} \cdot C_{\mathcal{Q},\Phi,p} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \sum_{\ell \in i^*} \left\| (c_j)_{j \in J_0 \cap J_\ell} \right\|_{\ell^2} \\ &= C_5 \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot (\Gamma_{\mathcal{Q}}\varrho)_i, \end{aligned}$$

with $C_5 := C_1 C_2 C_3 C_4^{1/p} \cdot C_{\mathcal{Q},\Phi,p}$. Since the right-hand side is an element of Y , the solidity of Y implies $g^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$, with

$$\begin{aligned} \|g^{(r)}\|_{\mathcal{D}_{\mathcal{F},\Phi}(\mathcal{Q}, L^p, Y)} &= \left\| \left(\left\| \mathcal{F}^{-1}(\varphi_i \cdot g^{(r)}) \right\|_{L^p} \right)_{i \in I} \right\|_Y \\ &\leq C_5 \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \|\Gamma_{\mathcal{Q}}\varrho\|_Y \\ &\leq C_5 \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \|\varrho\|_Y \\ &= C_6 \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \|c\|_V < \infty, \end{aligned} \quad (6.60)$$

where $C_6 := C_5 \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}$.

As seen above, we have now shown $g^{(r)} \in \mathcal{D}_K^{\mathcal{Q},p,Y}$. Since ι is well-defined and bounded, we conclude $g^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z)$, with

$$\begin{aligned} \|g^{(r)}\|_{\mathcal{D}_{\mathcal{F},\Psi}(\mathcal{P}, L^p, Z)} &\leq \|\iota\| \cdot \|g^{(r)}\|_{\mathcal{D}_{\mathcal{F},\Phi}(\mathcal{Q}, L^p, Y)} \\ &\leq C_6 \|\iota\| \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \|c\|_V. \end{aligned}$$

It remains to obtain a lower bound for $\|g^{(r)}\|_{\mathcal{D}_{\mathcal{F},\Psi}(\mathcal{P}, L^p, Z)}$. But this is a direct consequence of Corollary 6.4 (with \mathcal{P} instead of \mathcal{Q} and with $I_0 = M \cap J_{(r)}$): By choice of $J_{(r)}$, we have for $j, \ell \in M \cap J_{(r)}$ with $j \neq \ell$ that $P_j^* \cap P_\ell^* = \emptyset$. Furthermore, we have $(c_j)_{j \in M \cap J_{(r)}} \in \ell_0(M \cap J_{(r)})$, simply because M is finite. Finally, $\gamma_j \in C_c^\infty(\mathcal{O}')$ with $\text{supp } \gamma_j \subset P_j$ for all $j \in M \supset M \cap J_{(r)}$. Thus, Corollary 6.4 shows

$$C_1 \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot (c_j)_{j \in M \cap J_{(r)}} \stackrel{\text{eq. (6.54)}}{=} (c_j \cdot \|\mathcal{F}^{-1}\gamma_j\|_{L^p})_{j \in M \cap J_{(r)}} \in Z|_{M \cap J_{(r)}}$$

and yields a constant $C_7 = C_7(\mathcal{P}, p, d, C_{\mathcal{P}, \Psi, p}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}) > 0$ satisfying

$$\begin{aligned} \|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{P}, L^p, Z)} &\geq C_7^{-1} \cdot \left\| (c_j \cdot \|\mathcal{F}^{-1} \gamma_j\|_{L^p})_{j \in M \cap J_{(r)}} \right\|_{Z|_{M \cap J_{(r)}}} \\ \text{(eq. (6.54) and } c_j &= 0 \text{ for } j \in J_0 \setminus M) = C_1 \varepsilon^{d(1-\frac{1}{p})} C_7^{-1} \cdot \left\| (c_j \cdot \mathbf{1}_{J_0 \cap J_{(r)}}(j))_{j \in J_0} \right\|_{Z|_{J_0}}. \end{aligned}$$

Finally, using $J_0 = \biguplus_{r=1}^{r_0} (J_0 \cap J_{(r)})$ and the triangle inequality for Z , we arrive at $c \in Z|_{J_0}$ with

$$\begin{aligned} \|c\|_{Z|_{J_0}} &\leq C_8 \cdot \sum_{r=1}^{r_0} \|c \cdot \mathbf{1}_{J_0 \cap J_{(r)}}\|_{Z|_{J_0}} \\ &\leq C_8 \cdot \sum_{r=1}^{r_0} \frac{C_7}{C_1 \varepsilon^{d(1-\frac{1}{p})}} \|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{P}, L^p, Z)} \\ &\leq C_8 \cdot \sum_{r=1}^{r_0} \frac{C_7}{C_1 \varepsilon^{d(1-\frac{1}{p})}} C_6 \|\iota\| \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \|c\|_V \\ &= \frac{C_6 C_7 C_8 r_0}{C_1} \cdot \|c\|_V. \end{aligned}$$

for a suitable constant $C_8 = C_8(C_Z, r_0) = C_8(C_Z, N_{\mathcal{P}})$. This is precisely the desired embedding, since $c \in V_0$ was arbitrary.

Part 2: Here, we have $p \geq 2$ and $s = \max\{2, p\} = p$. The proof in this case is a mixture of that of Theorem 6.13 and of that of the first part. We only provide it here for the sake of completeness.

Corollary 6.6 (with $M \cap J_{(r)}$ in place of M and with $f_j = c_j \cdot \gamma_j$ for $j \in M \cap J_{(r)}$) yields a family $z = (z_j)_{j \in M \cap J_{(r)}} \in (\mathbb{R}^d)^{M \cap J_{(r)}}$ of modulations (possibly depending on c) satisfying

$$\begin{aligned} \left\| \mathcal{F}^{-1} \left[\sum_{j \in S} M_{z_j} (c_j \cdot \gamma_j) \right] \right\|_{L^p} &\leq 2 \cdot \left\| (\|\mathcal{F}^{-1} (c_j \cdot \gamma_j)\|_{L^p})_{j \in S} \right\|_{\ell^p} \\ \text{(eq. (6.54))} &= 2C_1 \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \left\| (c_j)_{j \in S} \right\|_{\ell^p} \end{aligned} \quad (6.61)$$

for all subsets $S \subset M \cap J_{(r)}$. With this choice of z , define

$$g^{(r)} := \sum_{j \in M \cap J_{(r)}} M_{z_j} (c_j \cdot \gamma_j).$$

As above, we have $\text{supp } g^{(r)} \subset K$ and hence $g^{(r)} \in \mathcal{D}_K^{\mathcal{Q}, p, Y}$, once we have shown $g^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$.

To see $g^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$, define $\varrho = (\varrho_\ell)_{\ell \in I}$ by $\varrho_\ell := \left\| (c_j)_{j \in J_0 \cap J_\ell} \right\|_{\ell^p}$. Note that we have $\varrho \in Y$ with $\|\varrho\|_Y = \|c\|_V$, since $c \in V = Y([\ell^s(J_0 \cap J_i)]_{i \in I})$ and because of $s = p$, since $p \geq 2$. Since Y is \mathcal{Q} -regular, this also implies $\Gamma_{\mathcal{Q}} \varrho \in Y$, which will soon become helpful.

Indeed, let $i \in I$ be arbitrary. Using estimate (6.56) (with $d_j = c_j \cdot \mathbf{1}_{M \cap J_{(r)}}(j)$), we derive

$$\begin{aligned} \left\| \mathcal{F}^{-1} (\varphi_i \cdot g^{(r)}) \right\|_{L^p} &\leq C_2 C_3 C_{\mathcal{Q}, \Phi, p} \cdot \sum_{\ell \in i^*} \left\| \mathcal{F}^{-1} \left[\sum_{j \in M^{(\ell)} \cap J_{(r)}} c_j \cdot \gamma_j \right] \right\|_{L^p} \\ \text{(eq. (6.61))} &\leq 2C_1 C_2 C_3 \cdot C_{\mathcal{Q}, \Phi, p} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \sum_{\ell \in i^*} \left\| (c_j)_{j \in M^{(\ell)} \cap J_{(r)}} \right\|_{\ell^p} \\ &\stackrel{(*)}{\leq} 2C_1 C_2 C_3 \cdot C_{\mathcal{Q}, \Phi, p} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \sum_{\ell \in i^*} \left\| (c_j)_{j \in J_0 \cap J_\ell} \right\|_{\ell^p} \\ &= C_9 \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot (\Gamma_{\mathcal{Q}} \varrho)_i, \end{aligned} \quad (6.62)$$

with $C_9 := 2C_1 C_2 C_3 \cdot C_{\mathcal{Q}, \Phi, p}$. Here, the step marked with $(*)$ used that we have $\xi_j \in P_j \cap Q_{\ell_j} = P_j \cap Q_\ell$ and hence $j \in J_0 \cap J_\ell$ for $j \in M^{(\ell)} \subset M \subset J_0$.

Since the right-hand side of estimate (6.62) lies in Y , the solidity of Y implies $g^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$, with

$$\begin{aligned} \|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)} &= \left\| \left(\left\| \mathcal{F}^{-1} \left(\varphi_i \cdot g^{(r)} \right) \right\|_{L^p} \right)_{i \in I} \right\|_Y \\ &\leq C_9 \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \|\Gamma_{\mathcal{Q}} \varrho\|_Y \\ &\leq C_9 \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \|\varrho\|_Y \\ &= C_{10} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \|c\|_V < \infty, \end{aligned}$$

where $C_{10} := C_9 \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}$. This is the exact analog of equation (6.60) in the first part of the proof. The remainder of the proof is now essentially identical to that of the first part and hence omitted. Simply note that Lemma 6.3 allows for an arbitrary modulation of the individual summands.

The final claim of the theorem—in case Z satisfies the Fatou property—is an easy consequence of Lemmas 4.5 and 4.6. For more details, see the proof of Theorem 6.15. \square

Now, in our final necessary criterion in this subsection, we will assume \mathcal{Q} (or a subfamily of \mathcal{Q}) to be almost subordinate to \mathcal{P} . Note though that no additional assumptions—like tightness—are necessary.

Theorem 6.18. Let $\emptyset \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p \in (0, \infty]$ and let $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ be two L^p -decomposition coverings of \mathcal{O} and \mathcal{O}' , respectively. Finally, let $Y \leq \mathbb{C}^I$ be \mathcal{Q} -regular and let $Z \leq \mathbb{C}^J$ be \mathcal{P} -regular.

Choose a subset $I_0 \subset I$ such that we have $Q_i^\circ \neq \emptyset$ for all $i \in I_0$ and such that the restricted family $\mathcal{Q}_{I_0} = (Q_i)_{i \in I_0}$ is almost subordinate to \mathcal{P} , i.e. there is $k \in \mathbb{N}_0$ such that for each $i \in I_0$ the inclusion $Q_i \subset P_{j_i}^{k*}$ is valid for a suitable $j_i \in J$. Define

$$K := \bigcup_{i \in I_0} Q_i \subset \mathcal{O} \cap \mathcal{O}'$$

and assume that the map

$$\iota : \left(\mathcal{D}_K^{\mathcal{Q}, p, Y}, \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)} \right) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z), f \mapsto f \quad (6.63)$$

is well-defined and continuous.

Let

$$s := \begin{cases} 1, & \text{if } p = \infty, \\ \min\{2, p\}, & \text{if } p < \infty \end{cases} = \begin{cases} 1, & \text{if } p = \infty, \\ p, & \text{if } 0 < p \leq 2, \\ 2, & \text{if } 2 < p < \infty. \end{cases} \quad (6.64)$$

Then the embedding

$$\eta : \ell_0(I_0) \cap Y|_{I_0} \hookrightarrow Z \left([\ell^s(I_0 \cap I_j)]_{j \in J} \right)$$

is well-defined and bounded, with $\|\eta\| \leq C \cdot \|\iota\|$ for some constant

$$C = C(d, p, k, C_Z, \mathcal{Q}, \mathcal{P}, C_{\mathcal{Q}, \Phi, p}, C_{\mathcal{P}, \Psi, p}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}) > 0.$$

Here, the L^p -BAPUs $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_j)_{j \in J}$ are those which are used to compute the (quasi)-norms on the respective decomposition spaces when computing the norm $\|\iota\|$.

Finally, if Z satisfies the Fatou property, the same statement as above also holds for the embedding $\eta : Y|_{I_0} \hookrightarrow Z \left([\ell^s(I_0 \cap I_j)]_{j \in J} \right)$, i.e. without restricting to $\ell_0(I_0)$. \blacktriangleleft

Remark. Note that we always have $s \leq 2$ and $s \leq p$. Thus, we have norm-decreasing embeddings $\ell^s(I_0 \cap I_j) \hookrightarrow \ell^2(I_0 \cap I_j)$ and $\ell^s(I_0 \cap I_j) \hookrightarrow \ell^p(I_0 \cap I_j)$. Consequently, it is easy to see that the theorem implies boundedness of the embeddings

$$\begin{aligned} \eta_1 : \ell_0(I_0) \cap Y|_{I_0} &\hookrightarrow Z \left([\ell^2(I_0 \cap I_j)]_{j \in J} \right), \\ \eta_2 : \ell_0(I_0) \cap Y|_{I_0} &\hookrightarrow Z \left([\ell^p(I_0 \cap I_j)]_{j \in J} \right), \end{aligned}$$

with $\|\eta_\ell\| \leq \|\eta\|$ for $\ell \in \underline{2}$, even without restricting to $\ell_0(I_0)$ if Z satisfies the Fatou property. \blacklozenge

We remark that parts of the following proof are very similar to parts of the proofs of Theorems 6.15 and 6.17. For the sake of completeness—and to make the proof self-contained—we still provide almost all details.

Proof. For brevity, set $V := Z\left([\ell^s(I_0 \cap I_j)]_{j \in J}\right)$. For arbitrary $i \in I_0$, we have $Q_i \subset P_{j_i}^{k*} \subset \mathcal{O}'$. Furthermore, $Q_i^\circ \neq \emptyset$ by assumption so that we get $Q_i \cap P_j \neq \emptyset$ for some $j \in j_i^{k*} \subset J$. In particular, $i \in I_j \cap I_0$ and thus $I_0 = \bigcup_{j \in J} (I_j \cap I_0)$. All in all, this shows that $V \leq \mathbb{C}^{I_0}$ is a solid sequence space over I_0 .

Fix a nonzero, nonnegative $\gamma \in C_c^\infty(B_1(0))$ for the rest of the proof. Set $r_0 := N_{\mathcal{P}}^{2(2k+3)+1} = N_{\mathcal{P}}^{4k+7}$ and note that Lemma 2.14 yields a partition $J = \bigsqcup_{r=1}^{r_0} J^{(r)}$ such that $P_j^{(2k+3)*} \cap P_\ell^{(2k+3)*} = \emptyset$ holds for all $j, \ell \in J^{(r)}$ with $j \neq \ell$ and all $r \in \underline{r_0}$.

As a crucial consequence of this, we have

$$\forall j, \ell \in J^{(r)} \text{ with } j \neq \ell : \quad (I_j \cap I_0) \cap (I_\ell \cap I_0) = \emptyset, \quad (6.65)$$

because for $i \in I_j \cap I_\ell \cap I_0$, we would have $Q_i \cap P_j \neq \emptyset \neq Q_i \cap P_\ell$, so that Lemma 2.11 would yield

$$\emptyset \neq Q_i \subset P_j^{(2k+2)*} \cap P_\ell^{(2k+2)*} \subset P_j^{(2k+3)*} \cap P_\ell^{(2k+3)*}$$

in contradiction to $j, \ell \in J^{(r)}$ with $j \neq \ell$.

These considerations show that

$$I^{(r)} := \bigcup_{j \in J^{(r)}} (I_j \cap I_0) \subset I_0$$

is well-defined. As a simple consequence of this definition, we observe

$$\forall j \in J^{(r)} : \quad I^{(r)} \cap I_j = I_0 \cap I_j. \quad (6.66)$$

Now, let $c = (c_i)_{i \in I_0} \in \ell_0(I_0) \cap Y|_{I_0}$ be arbitrary and set $M := \text{supp } c \subset I_0$. Since each Q_i with $i \in M \subset I_0$ has nonempty interior and because M is finite, there is some $\varepsilon = \varepsilon(c) > 0$ and for each $i \in M$ some $\xi_i \in \mathbb{R}^d$ (possibly depending on c) with $B_\varepsilon(\xi_i) \subset Q_i$. Note that although ε and the family $(\xi_i)_{i \in M}$ may depend heavily on the specific sequence c —more precisely on its support—none of the constants C_1, C_2, \dots in this proof will depend on c .

For $i \in M$, define $\gamma_i := L_{\xi_i}[\gamma(\varepsilon^{-1} \cdot)]$ as in the proof of Theorem 6.17 and note that equations (6.53) and (6.54) (with $C_1 := \|\mathcal{F}^{-1}\gamma\|_{L^p}$, i.e. $C_1 = C_1(p, d)$) still apply, with j replaced by i . Fix some $r \in \underline{r_0}$ (almost) until the end of the proof.

In the following, we will “test” the embedding ι with a function of the form

$$g_{z, \theta}^{(r)} := \sum_{i \in M \cap I^{(r)}} M_{z_i} [\theta_i |c_i| \cdot \gamma_i]$$

for suitable modulations $z = (z_i)_{i \in M \cap I^{(r)}} \in (\mathbb{R}^d)^{M \cap I^{(r)}}$ and signs $\theta = (\theta_i)_{i \in M \cap I^{(r)}} \in \{\pm 1\}^{M \cap I^{(r)}}$. To this end, first note that equation (6.54) and the solidity of Y yield

$$(|c_i| \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p})_{i \in I} = \left(C_1 \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot |c_i|\right)_{i \in I} \in Y,$$

so that Lemma 6.2 (with $k = 0$ and $c_i = \gamma_i = 0$ for $i \in I \setminus M$) implies $g_{z, \theta}^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$ with

$$\begin{aligned} \|g_{z, \theta}^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)} &\leq C_2 \cdot \left\| (|c_i| \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p})_{i \in M \cap I^{(r)}} \right\|_{Y|_{M \cap I^{(r)}}} \\ &= C_1 C_2 \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \left\| (|c_i|)_{i \in M \cap I^{(r)}} \right\|_{Y|_{M \cap I^{(r)}}} \\ (Y \text{ solid}) &\leq C_1 C_2 \varepsilon^{d(1-\frac{1}{p})} \cdot \|c\|_{Y|_{I_0}} < \infty \end{aligned}$$

for some constant $C_2 = C_2(d, p, \mathcal{Q}, C_{\mathcal{Q}, \Phi, p}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y})$. Here, we used $\text{supp } \gamma_i \subset B_\varepsilon(\xi_i) \subset Q_i$ for all $i \in M \cap I^{(r)}$.

Since we have $\gamma_i \in C_c^\infty(\mathcal{O})$ with $\text{supp } \gamma_i \subset Q_i \subset K$ for all $i \in I_0$ and since the sum defining $g_{z,\theta}^{(r)}$ is finite, we get $g_{z,\theta}^{(r)} \in C_c^\infty(\mathcal{O})$ with $\text{supp } g_{z,\theta}^{(r)} \subset K$ and thus $g_{z,\theta}^{(r)} \in \mathcal{D}_K^{\mathcal{Q},p,Y}$. Since ι is well-defined and bounded, this implies $g_{z,\theta}^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^p, Z)$, with

$$\begin{aligned} \|g_{z,\theta}^{(r)}\|_{\mathcal{D}_{\mathcal{F},\Psi}(\mathcal{P}, L^p, Z)} &\leq \|\iota\| \cdot \|g_{z,\theta}^{(r)}\|_{\mathcal{D}_{\mathcal{F},\Phi}(\mathcal{Q}, L^p, Y)} \\ &\leq C_1 C_2 \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \|\iota\| \cdot \|c\|_{Y|_{I_0}} < \infty. \end{aligned} \quad (6.67)$$

In the remainder of the proof, we will obtain lower bounds on $\|g_{z,\theta}^{(r)}\|_{\mathcal{D}_{\mathcal{F},\Psi}(\mathcal{P}, L^p, Z)}$, for suitable values of z and θ .

To this end, we will now prove the following observation: For arbitrary $j \in J^{(r)}$ and $i \in M \cap I^{(r)}$, we have

$$\psi_j^{(2k+3)*} \cdot \gamma_i = \begin{cases} \gamma_i, & \text{if } i \in I_j, \\ 0, & \text{if } i \notin I_j. \end{cases} \quad (6.68)$$

Indeed, for $i \in M \subset I_0$ and $j \in J$ with $Q_i \cap P_j \neq \emptyset$, Lemma 2.11 yields

$$\text{supp } \gamma_i \subset Q_i \subset P_j^{(2k+2)*}$$

and Lemma 2.4 implies $\psi_j^{(2k+3)*} \equiv 1$ on $P_j^{(2k+2)*}$. We have thus established the following:

$$\forall i \in M \forall j \in J \text{ with } Q_i \cap P_j \neq \emptyset : \quad \psi_j^{(2k+3)*} \gamma_i = \gamma_i. \quad (6.69)$$

Now, let $j \in J^{(r)}$ and $i \in M \cap I^{(r)} \subset I_0$ be arbitrary. In case of $i \in I_j$, equation (6.69) yields $\psi_j^{(2k+3)*} \gamma_i = \gamma_i$. Otherwise, if $i \notin I_j$, then the definition of $I^{(r)}$ yields some $\ell \in J^{(r)} \setminus \{j\}$ with $i \in I_\ell \cap I_0$. Note that this entails $\psi_\ell^{(2k+3)*} \gamma_i = \gamma_i$ by equation (6.69). But because of $j, \ell \in J^{(r)}$ with $j \neq \ell$, we get $P_j^{(2k+3)*} \cap P_\ell^{(2k+3)*} = \emptyset$ which entails $\psi_j^{(2k+3)*} \psi_\ell^{(2k+3)*} \equiv 0$ and thus

$$\psi_j^{(2k+3)*} \cdot \gamma_i = \psi_j^{(2k+3)*} \cdot \psi_\ell^{(2k+3)*} \gamma_i \equiv 0.$$

All in all, we have established identity (6.68) from above in both cases.

Next, Remark 3.16 shows that the family $\Gamma = \left(\psi_j^{(2k+3)*} \right)_{j \in J}$ is an L^p -bounded family for \mathcal{P} , with $C_{\mathcal{P}, \Gamma, p} \leq C_3 = C_3(\mathcal{P}, C_{\mathcal{P}, \Psi, p}, d, p, k)$ and $\ell_{\Gamma, \mathcal{P}} = 2k+3$. Consequently, Theorem 3.17 yields a further constant $C_4 = C_4(\mathcal{P}, p, d, k, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}) > 0$ satisfying

$$\begin{aligned} \left\| \left(\left\| \mathcal{F}^{-1} \left(\psi_j^{(2k+3)*} \cdot g_{z,\theta}^{(r)} \right) \right\|_{L^p} \right)_{j \in J^{(r)}} \right\|_{Z|_{J^{(r)}}} &\leq \left\| \left(\left\| \mathcal{F}^{-1} \left(\psi_j^{(2k+3)*} \cdot g_{z,\theta}^{(r)} \right) \right\|_{L^p} \right)_{j \in J} \right\|_Z \\ &\leq C_4 \cdot C_{\mathcal{P}, \Gamma, p} \cdot \|g_{z,\theta}^{(r)}\|_{\mathcal{D}_{\mathcal{F},\Psi}(\mathcal{P}, L^p, Z)} \\ (\text{eq. (6.67)}) &\leq C_1 C_2 C_3 C_4 \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \|\iota\| \cdot \|c\|_{Y|_{I_0}}. \end{aligned}$$

But equation (6.68) and the special form of $g_{z,\theta}^{(r)}$ yield for $j \in J^{(r)}$ that

$$\begin{aligned} \psi_j^{(2k+3)*} \cdot g_{z,\theta}^{(r)} &= \sum_{i \in M \cap I^{(r)}} \left[\theta_i |c_i| \cdot M_{z_i} \left(\psi_j^{(2k+3)*} \cdot \gamma_i \right) \right] \\ &= \sum_{i \in M \cap I^{(r)} \cap I_j} M_{z_i} [\theta_i |c_i| \cdot \gamma_i], \end{aligned}$$

so that we get—for $C_5 := C_1 C_2 C_3 C_4$ —the estimate

$$\left\| \left(\left\| \mathcal{F}^{-1} \left(\sum_{i \in M \cap I^{(r)} \cap I_j} M_{z_i} [\theta_i |c_i| \cdot \gamma_i] \right) \right\|_{L^p} \right)_{j \in J^{(r)}} \right\|_{Z|_{J^{(r)}}} \leq C_5 \cdot \varepsilon^{d(1-\frac{1}{p})} \|\iota\| \cdot \|c\|_{Y|_{I_0}} \quad (6.70)$$

for all $r \in \underline{r_0}$ and $z \in (\mathbb{R}^d)^{M \cap I^{(r)}}$, as well as $\theta \in \{\pm 1\}^{M \cap I^{(r)}}$.

In the remainder of the proof, we will show for each $r \in \underline{r_0}$ that one can choose the parameters $z^{(r)} = (z_i^{(r)})_{i \in M \cap I^{(r)}} \in (\mathbb{R}^d)^{M \cap I^{(r)}}$ and $\theta^{(r)} = (\theta_i^{(r)})_{i \in M \cap I^{(r)}} \in \{\pm 1\}^{M \cap I^{(r)}}$ in such a way that we get

$$\varepsilon^{d(1-\frac{1}{p})} \cdot \|(c_i)_{i \in I_0 \cap I_j}\|_{\ell^s} \leq C_6 \cdot \left\| \mathcal{F}^{-1} \left(\sum_{i \in M \cap I_j \cap I^{(r)}} M_{z_i^{(r)}} \left[\theta_i^{(r)} |c_i| \cdot \gamma_i \right] \right) \right\|_{L^p} \quad \forall j \in J^{(r)} \quad (6.71)$$

for a suitable constant $C_6 = C_6(d, p)$. Once this is done, the quasi-triangle inequality for Z and the identity $J = \biguplus_{r=1}^{r_0} J^{(r)}$ yield a constant $C_7 = C_7(C_Z, r_0) = C_7(C_Z, k, N_{\mathcal{P}})$ satisfying

$$\begin{aligned} \varepsilon^{d(1-\frac{1}{p})} \cdot \|c\|_V &= \left\| \left(\varepsilon^{d(1-\frac{1}{p})} \cdot \|(c_i)_{i \in I_0 \cap I_j}\|_{\ell^s} \right)_{j \in J} \right\|_Z \\ &\leq C_7 \cdot \sum_{r=1}^{r_0} \left\| \left(\varepsilon^{d(1-\frac{1}{p})} \cdot \|(c_i)_{i \in I_0 \cap I_j}\|_{\ell^s} \right)_{j \in J^{(r)}} \right\|_{Z|_{J^{(r)}}} \\ &\leq C_6 C_7 \cdot \sum_{r=1}^{r_0} \left\| \left(\left\| \mathcal{F}^{-1} \left(\sum_{i \in M \cap I_j \cap I^{(r)}} M_{z_i^{(r)}} \left[\theta_i^{(r)} |c_i| \cdot \gamma_i \right] \right) \right\|_{L^p} \right)_{j \in J^{(r)}} \right\|_{Z|_{J^{(r)}}} \\ &\leq r_0 C_5 C_6 C_7 \cdot \varepsilon^{d(1-\frac{1}{p})} \|\iota\| \cdot \|c\|_{Y|_{I_0}}, \end{aligned}$$

so that canceling the common factor $\varepsilon^{d(1-\frac{1}{p})}$ yields the claim. Note that the quantitative arguments from above (and solidity of Z) in particular yield the qualitative conclusion $c \in V$.

It remains to show that we can choose $z^{(r)}, \theta^{(r)}$ in such a way that estimate (6.71) is fulfilled. To this end we will distinguish the three cases which are indicated by the definition of s , cf. equation (6.64).

Case 1: $p = \infty$. In this case, choose $z_i^{(r)} = 0$ and $\theta_i^{(r)} = 1$ for all $i \in M \cap I^{(r)}$. Then the continuity of $\mathcal{F}^{-1} \left(\sum_{i \in M \cap I_j} |c_i| \gamma_i \right) \in \mathcal{S}(\mathbb{R}^d)$ implies because of $M \cap I_j = M \cap I_0 \cap I_j = M \cap I_j \cap I^{(r)}$ (cf. equation (6.66)) that

$$\begin{aligned} \left\| \mathcal{F}^{-1} \left(\sum_{i \in M \cap I_j \cap I^{(r)}} M_{z_i^{(r)}} \left[\theta_i^{(r)} |c_i| \cdot \gamma_i \right] \right) \right\|_{L^p} &= \left\| \mathcal{F}^{-1} \left(\sum_{i \in M \cap I_j} |c_i| \cdot \gamma_i \right) \right\|_{L^\infty} \\ &\geq \left| \sum_{i \in M \cap I_j} (\mathcal{F}^{-1} [|c_i| \cdot \gamma_i]) (0) \right| \\ &= \left| \sum_{i \in M \cap I_j} \int_{\mathbb{R}^d} |c_i| \cdot \gamma_i(\xi) \, d\xi \right| \\ (\gamma_i \geq 0) &= \sum_{i \in M \cap I_j} |c_i| \cdot \|\gamma_i\|_{L^1} \\ (\text{Riemann-Lebesgue}) &\geq \sum_{i \in M \cap I_j} |c_i| \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^\infty} \\ (\text{eq. (6.54) and } c_i = 0 \text{ if } i \in I_0 \setminus M) &= C_1 \varepsilon^{d(1-\frac{1}{p})} \cdot \|(c_i)_{i \in I_0 \cap I_j}\|_{\ell^1}. \end{aligned}$$

for all $j \in J^{(r)}$. Since we have $s = 1$ for $p = \infty$, this is nothing but estimate (6.71), with $C_6 = C_1^{-1}$.

Case 2: $2 < p < \infty$. For $j \in J^{(r)}$ consider the random variable $\omega^{(j)} = (\omega_i^{(j)})_{i \in M \cap I_j} \in \{\pm 1\}^{M \cap I_j}$ which we take to be uniformly distributed in $\{\pm 1\}^{M \cap I_j}$. Let $C_8 = C_8(p) > 0$ be the constant provided by Khintchine's inequality (Theorem 6.7). By applying that inequality (marked with (\dagger)), as well as

equation (6.53)—i.e. $\mathcal{F}^{-1}\gamma_i = \varepsilon^d \cdot M_{\xi_i}[(\mathcal{F}^{-1}\gamma)(\varepsilon \cdot)]$ (marked with (\ddagger)), we conclude

$$\begin{aligned}
 \mathbb{E} \left\| \mathcal{F}^{-1} \left[\sum_{i \in I_j \cap M} |c_i| \omega_i^{(j)} \gamma_i \right] \right\|_{L^p}^p &= \mathbb{E} \int_{\mathbb{R}^d} \left| \sum_{i \in I_j \cap M} |c_i| \omega_i^{(j)} \cdot (\mathcal{F}^{-1}\gamma_i)(x) \right|^p dx \\
 &\stackrel{(\ddagger)}{=} \varepsilon^{dp} \cdot \int_{\mathbb{R}^d} \mathbb{E} \left| \sum_{i \in I_j \cap M} |c_i| \omega_i^{(j)} \cdot e^{2\pi i \langle \xi_i, x \rangle} \cdot (\mathcal{F}^{-1}\gamma)(\varepsilon x) \right|^p dx \\
 &\stackrel{(\dagger)}{\geq} C_8^{-1} \varepsilon^{dp} \cdot \int_{\mathbb{R}^d} \left(\sum_{i \in I_j \cap M} \left| |c_i| \cdot e^{2\pi i \langle \xi_i, x \rangle} \cdot (\mathcal{F}^{-1}\gamma)(\varepsilon x) \right|^2 \right)^{p/2} dx \\
 &= C_8^{-1} \varepsilon^{dp} \cdot \left(\sum_{i \in I_j \cap M} |c_i|^2 \right)^{p/2} \cdot \int_{\mathbb{R}^d} |(\mathcal{F}^{-1}\gamma)(\varepsilon x)|^p dx \\
 &= \frac{\|\mathcal{F}^{-1}\gamma\|_{L^p}^p}{C_8} \cdot \varepsilon^{d(p-1)} \cdot \|(c_i)_{i \in I_j \cap M}\|_{\ell^2}^p \\
 &= \left[\frac{\|\mathcal{F}^{-1}\gamma\|_{L^p}}{C_8^{1/p}} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \|(c_i)_{i \in I_0 \cap I_j}\|_{\ell^2} \right]^p.
 \end{aligned}$$

Here, we used $c_i = 0$ for $i \in I_0 \setminus M$ in the last step.

In particular, the estimate above yields a *deterministic* realization $\omega^{(j)} = (\omega_i^{(j)})_{i \in I_j \cap M} \in \{\pm 1\}^{I_j \cap M}$ with

$$\left\| \mathcal{F}^{-1} \left[\sum_{i \in I_j \cap M} |c_i| \omega_i^{(j)} \gamma_i \right] \right\|_{L^p} \geq \frac{\|\mathcal{F}^{-1}\gamma\|_{L^p}}{C_8^{1/p}} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \|(c_i)_{i \in I_0 \cap I_j}\|_{\ell^2}. \quad (6.72)$$

Recall from above that the sets $(I_0 \cap I_j)_{j \in J^{(r)}}$ form a partition of the set $I^{(r)} \subset I_0$. Since $M \subset I_0$, we thus see that $(I_j \cap M)_{j \in J^{(r)}}$ partitions $M \cap I^{(r)}$. Thus, we can combine the individual “sign realizations” $\omega^{(j)}$ for $j \in J^{(r)}$ into the “global” sequence $\theta^{(r)} = (\theta_i^{(r)})_{i \in M \cap I^{(r)}}$ given by $\theta_i^{(r)} = \omega_i^{(j)}$, where—for given $i \in M \cap I^{(r)}$ —the index $j = j_i$ is the unique $j \in J^{(r)}$ with $i \in M \cap I_j$.

Note that we have $\theta_i^{(r)} = \omega_i^{(j)}$ for all $i \in M \cap I_j$ and arbitrary $j \in J^{(r)}$. Furthermore, recall from equation (6.66) that $M \cap I_j \cap I^{(r)} = M \cap I_0 \cap I_j = M \cap I_j$. Thus, given this “joint sign choice” $\theta^{(r)}$, our previous considerations easily imply that equation (6.71) is indeed satisfied, with $z_i^{(r)} = 0$ for all $i \in M \cap I^{(r)}$ and with $C_6 = C_8^{1/p} \cdot \|\mathcal{F}^{-1}\gamma\|_{L^p}^{-1}$, since we have $s = 2$ in the present case $2 < p < \infty$.

Case 3: We have $0 < p \leq 2$. In this case, we choose $\theta_i^{(r)} = 1$ for all $i \in M$. Next, Corollary 6.6, applied to the family $(f_i)_{i \in M \cap I^{(r)}}$ with $f_i := |c_i| \gamma_i \in \mathcal{S}(\mathbb{R}^d)$, yields a family $z^{(r)} = (z_i^{(r)})_{i \in M \cap I^{(r)}}$ of modulations satisfying

$$\begin{aligned}
 \left\| \mathcal{F}^{-1} \left[\sum_{i \in S} M_{z_i^{(r)}} \left(\theta_i^{(r)} |c_i| \cdot \gamma_i \right) \right] \right\|_{L^p} &= \left\| \mathcal{F}^{-1} \left[\sum_{i \in S} M_{z_i^{(r)}} f_i \right] \right\|_{L^p} \\
 &\geq \frac{1}{2} \cdot \left\| (\|\mathcal{F}^{-1} f_i\|_{L^p})_{i \in S} \right\|_{\ell^p} \\
 &= \frac{1}{2} \cdot \left\| (|c_i| \cdot \|\mathcal{F}^{-1} \gamma_i\|_{L^p})_{i \in S} \right\|_{\ell^p} \\
 (\text{eq. (6.54)}) &= \frac{C_1}{2} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \|(c_i)_{i \in S}\|_{\ell^p}
 \end{aligned}$$

for arbitrary subsets $S \subset M \cap I^{(r)}$.

If we apply this with $S = M \cap I_j = M \cap I_0 \cap I_j = M \cap I^{(r)} \cap I_j$ (for $j \in J^{(r)}$) and recall $s = p$ (because of $0 < p \leq 2$), we get

$$\left\| \mathcal{F}^{-1} \left[\sum_{i \in M \cap I_j \cap I^{(r)}} M_{z_i^{(r)}} \left(\theta_i^{(r)} |c_i| \cdot \gamma_i \right) \right] \right\|_{L^p} \geq \frac{C_1}{2} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \left\| (c_i)_{i \in M \cap I_j} \right\|_{\ell^p}$$

$$(s = p \text{ and } c_i = 0 \text{ for } i \in I_0 \setminus M) = \frac{C_1}{2} \cdot \varepsilon^{d(1-\frac{1}{p})} \cdot \left\| (c_i)_{i \in I_0 \cap I_j} \right\|_{\ell^s},$$

which is nothing but equation (6.71), with $C_6 = 2C_1^{-1}$.

This completes the proof, with the exception of the additional claim in case Z satisfies the Fatou property. But this claim is an easy consequence of Lemmas 4.5 and 4.6. For more details, see the proof of Theorem 6.15. \square

6.5. Complete characterizations for relatively moderate coverings. In this subsection, we develop further necessary conditions for embeddings between decomposition spaces. These will show that in general, one has to use the exponents p_2^∇ and p_1^Δ (or $p_1^{\pm\Delta}$) to calculate the “inner norm”, as in the sufficient conditions from Remarks 5.9 and 5.13. Note that up to now, all our necessary conditions were only able to show that these conditions are necessary for existence of the embedding with p_2^∇ replaced by p_2 and with p_1^Δ replaced by p_1 , cf. Theorems 6.15 and 6.13, with a slight improvement possible in case of $p_1 = p_2$, cf. Theorems 6.17 and 6.18.

For the proof of our stronger necessary conditions, we place more severe restrictions on the relation between the two coverings and on the “global” components of the two decomposition spaces. Precisely, we will only consider weighted Lebesgue sequence spaces as our global components. Furthermore, we will assume that one covering is almost subordinate as well as *relatively moderate* with respect to the other and that the weight of the “subordinate” covering is also moderate with respect to the “coarse” covering.

Note that in the preceding subsections, the functions which we used to “test” the embedding were always adapted to the “finer” of the two coverings. In contrast, in this section, our “test functions” will be adapted to the “coarser” of the two coverings. We remark that the basic idea of this construction is taken from Han and Wang[13], who use a similar construction for the special case of embeddings between α -modulation spaces.

As a preparation, we first establish a necessary condition which does *not* assume relative moderateness of the two coverings. Afterwards, we specialize this to the relatively moderate case. Occasionally, the following lemma—especially the ensuing remark—will also be useful for coverings which are not relatively moderate to each other.

Lemma 6.19. *Let $\emptyset \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ be a semi-structured L^{p_1} -decomposition covering of \mathcal{O} and let $\mathcal{P} = (P_j)_{j \in J} = (S_j P'_j + c_j)_{j \in J}$ be a tight semi-structured L^{p_2} -decomposition covering of \mathcal{O}' for certain $p_1, p_2 \in (0, \infty]$. Let $Y \leq \mathbb{C}^I$ and $Z \leq \mathbb{C}^J$ be \mathcal{Q} -regular and \mathcal{P} -regular, respectively.*

Let $J_0 \subset J$ such that $\mathcal{P}_{J_0} = (P_j)_{j \in J_0}$ is almost subordinate to \mathcal{Q} and define

$$I_0 := \{i \in I \mid J_0 \cap J_i \neq \emptyset\}$$

as well as $k := k(\mathcal{P}_{J_0}, \mathcal{Q})$ and set¹¹

$$K := \bigcup_{i \in I_0} \overline{Q_i^{(2k+3)*}} \subset \mathcal{O},$$

and assume that there is a bounded linear map

$$\iota : \left(\mathcal{D}_K^{\mathcal{Q}, p_1, Y}, \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)} \right) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$$

which satisfies $\langle \iota f, \varphi \rangle = \langle f, \varphi \rangle$ for all $\varphi \in C_c^\infty(\mathcal{O} \cap \mathcal{O}')$ and all $f \in \mathcal{D}_K^{\mathcal{Q}, p_1, Y}$.

¹¹The inclusion $K \subset \mathcal{O}$ is a direct consequence of Lemma 2.4.

Then the map

$$\eta : \ell_0(I_0) \cap Y|_{I_0} \rightarrow Z, (x_i)_{i \in I_0} \mapsto \sum_{i \in I_0} \left[x_i |\det T_i|^{\frac{1}{p_1}-1} \cdot \left(|\det S_j|^{1-\frac{1}{p_2}} \mathbf{1}_{J_0 \cap J_i}(j) \right)_{j \in J} \right]$$

is well-defined and bounded, with $\|\eta\| \leq C \cdot \|\iota\|$, for a constant $C > 0$ of the form

$$C = C(d, k, C_Z, p_1, p_2, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}).$$

Here, as usual, the L^{p_1} -BAPU $\Phi = (\varphi_i)_{i \in I}$ has to be used to calculate the (quasi)-norm on the decomposition space $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$ for computing $\|\iota\|$.

Finally, in case of $Z = \ell_v^{q_2}(J)$ for a \mathcal{P} -moderate weight $v = (v_j)_{j \in J}$ and some $q_2 \in (0, \infty]$, we get that the embedding

$$\gamma : Y|_{I_0} \hookrightarrow \ell_u^{q_2}(I_0) \quad \text{with} \quad u_i := |\det T_i|^{\frac{1}{p_1}-1} \cdot \left\| \left(v_j \cdot |\det S_j|^{1-\frac{1}{p_2}} \right)_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2}}$$

is well-defined and bounded, with $\|\gamma\| \leq C' \cdot \|\iota\|$ for some constant

$$C' = C'(d, k, q_2, p_1, p_2, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{P}}, C_{v, \mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}).$$

In particular, $u_i < \infty$ for all $i \in I_0$ with $\delta_i \in Y$. ◀

Remark 6.20. As we have seen previously, even applying the derived embeddings for sequence spaces to sequences which are supported on a single point can be useful. Indeed, in the present case, let $j_0 \in J_0$ be arbitrary and let $i_0 \in I$ with $Q_{i_0} \cap P_{j_0} \neq \emptyset$. This yields $j_0 \in J_0 \cap J_{i_0}$ and hence $i_0 \in I_0$. Now, if $\delta_{i_0} \in Y$, we can apply the lemma to conclude

$$|\det T_{i_0}|^{\frac{1}{p_1}-1} \cdot |\det S_{j_0}|^{1-\frac{1}{p_2}} \cdot \|\delta_{j_0}\|_Z \leq |\det T_{i_0}|^{\frac{1}{p_1}-1} \cdot \left\| \left(|\det S_j|^{1-\frac{1}{p_2}} \right)_{j \in J_0 \cap J_{i_0}} \right\|_{Z|_{J_0 \cap J_{i_0}}} \leq \|\eta\| \cdot \|\delta_{i_0}\|_Y.$$

In several cases, this estimate yields sharper conditions than estimate (6.35) from Remark 6.14, cf. the proof of Theorem 9.21 for an example where this occurs. ♦

Proof. Choose $(\gamma_j)_{j \in J}$ as in Lemma 6.12 (applied to \mathcal{P} instead of \mathcal{Q}). Let $r_0 := N_{\mathcal{Q}}^{2(2k+3)+1} = N_{\mathcal{Q}}^{4k+7}$, so that Lemma 2.14 yields a partition $I = \bigsqcup_{r=1}^{r_0} I^{(r)}$ for which $Q_i^{(2k+3)*} \cap Q_{\ell}^{(2k+3)*} = \emptyset$ holds for all $r \in \underline{r_0}$ and all $i, \ell \in I^{(r)}$ with $i \neq \ell$.

We first note that the assumption on ι implies

$$\gamma_j \cdot \iota f = \gamma_j \cdot f \text{ for all } f \in \mathcal{D}_{K}^{\mathcal{Q}, p_1, Y} \text{ and } j \in J_0, \quad (6.73)$$

where the equality has to be understood in the sense of tempered distributions. To see this, let $g \in \mathcal{S}(\mathbb{R}^d)$ be arbitrary. By definition, we have

$$\langle \gamma_j \cdot \iota f, g \rangle = \langle \iota f, \gamma_j g \rangle \stackrel{(\dagger)}{=} \langle f, \gamma_j g \rangle = \langle \gamma_j \cdot f, g \rangle,$$

where we used at (\dagger) that $\text{supp } \gamma_j \subset P_j \subset Q_{i_j}^{k*} \subset \mathcal{O}$ for some $i_j \in I$ and also $\text{supp } \gamma_j \subset P_j \subset \mathcal{O}'$. Together, we conclude $\gamma_j g \in C_c^\infty(\mathcal{O} \cap \mathcal{O}')$, so that $\langle \iota f, \gamma_j g \rangle = \langle f, \gamma_j g \rangle$ holds by assumption on ι .

For $r \in \underline{r_0}$, the family $(J_0 \cap J_i)_{i \in I^{(r)}}$ is pairwise disjoint, because for $i, \ell \in I^{(r)}$ with $j \in J_0 \cap J_i \cap J_\ell$, Lemma 2.11 would imply

$$\emptyset \neq P_j \subset Q_i^{(2k+2)*} \cap Q_\ell^{(2k+2)*} \subset Q_i^{(2k+3)*} \cap Q_\ell^{(2k+3)*}$$

which in turn yields $i = \ell$ by choice of $(I^{(r)})_{r \in \underline{r_0}}$. Thus, we can define

$$J^{(r)} := \bigsqcup_{i \in I^{(r)}} (J_0 \cap J_i) \subset J_0.$$

Note that this definition yields

$$J^{(r)} \cap J_i = J_0 \cap J_i \quad \forall i \in I^{(r)}.$$

As a next step, we note for $i \in I^{(r)}$ and $j \in J^{(r)}$ that

$$\gamma_j \varphi_i^{(2k+3)*} = \begin{cases} \gamma_j, & \text{if } j \in J_0 \cap J_i, \\ 0, & \text{otherwise.} \end{cases} \quad (6.74)$$

To see this, first assume $j \in J_0 \cap J_i$. Lemma 2.11 yields $P_j \subset Q_i^{(2k+2)*}$ and Lemma 2.4 implies $\varphi_i^{(2k+3)*} \equiv 1$ on $Q_i^{(2k+2)*} \supset P_j$. Since γ_j vanishes outside of P_j , this establishes the first case. For the case $j \notin J_0 \cap J_i$, observe that we have $j \in J^{(r)}$ and thus $j \in J_0 \cap J_\ell$ for some $\ell \in I^{(r)}$. As we just saw, this implies $\gamma_j = \gamma_j \varphi_\ell^{(2k+3)*}$. But $j \notin J_0 \cap J_i$ yields $i \neq \ell$ and thus $Q_i^{(2k+3)*} \cap Q_\ell^{(2k+3)*} = \emptyset$ because of $i, \ell \in I^{(r)}$. This leads to

$$\gamma_j \varphi_i^{(2k+3)*} = \gamma_j \varphi_\ell^{(2k+3)*} \varphi_j^{(2k+3)*} \equiv 0,$$

so that equation (6.74) is established.

Now, let $d = (d_i)_{i \in I_0} \in \ell_0(I_0) \cap Y|_{I_0}$ be arbitrary and consider d as an element of $\ell_0(I)$ by extending it trivially. Set $\zeta_i := |\det T_i|^{\frac{1}{p_1}-1} \cdot d_i$ for $i \in I$. Finally, for $r \in \underline{r}_0$, define

$$g^{(r)} := \sum_{i \in I_0 \cap I^{(r)}} \zeta_i \varphi_i^{(2k+3)*}$$

and note that we have $g^{(r)} \in C_c^\infty(\mathcal{O})$ with $\text{supp } g^{(r)} \subset K$, as a *finite* sum of functions which satisfy the same properties.

Now, note that Lemma 6.12 shows

$$\left\| \mathcal{F}^{-1} \varphi_i^{(2k+3)*} \right\|_{L^{p_1}} \leq C_1 \cdot |\det T_i|^{1-\frac{1}{p_1}}$$

for all $i \in I$ and a constant $C_1 = C_1(d, p_1, k, \mathcal{Q}, C_{\mathcal{Q}, \Phi, p_1})$. Thus, Lemma 6.2 implies $g^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)$, with

$$\begin{aligned} \|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)} &\leq C_2 \cdot \left\| \left(\zeta_i \cdot \left\| \mathcal{F}^{-1} \varphi_i^{(2k+3)*} \right\|_{L^{p_1}} \right)_{i \in I} \right\|_Y \\ &\leq C_1 C_2 \cdot \|d\|_Y = C_1 C_2 \cdot \|d\|_{Y|_{I_0}} < \infty \end{aligned} \quad (6.75)$$

for some constant $C_2 = C_2(k, d, p_1, \mathcal{Q}, C_{\mathcal{Q}, \Phi, p_1}, \|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y})$. All in all, we have shown $g^{(r)} \in \mathcal{D}_K^{\mathcal{Q}, p_1, Y}$, so that $\iota g^{(r)} \in \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)$ is well-defined.

Now, we want to establish a lower bound for $\|\iota g^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)}$. To this end, note that for $i \in I_0 \cap I^{(r)}$ and $j \in J_0 \cap J_i \subset J^{(r)}$, equations (6.74) and (6.73) and the pairwise disjointness of the family $(J_0 \cap J_\ell)_{\ell \in I^{(r)}}$ yield

$$\left\| \mathcal{F}^{-1}(\gamma_j \cdot \iota g^{(r)}) \right\|_{L^{p_2}} = \zeta_i \cdot \left\| \mathcal{F}^{-1} \gamma_j \right\|_{L^{p_2}} = C_3 \cdot \zeta_i \cdot |\det S_j|^{1-\frac{1}{p_2}} \quad \forall i \in I_0 \cap I^{(r)} \text{ and } j \in J_0 \cap J_i \quad (6.76)$$

where the constant $C_3 = C_3(d, p_2, \varepsilon_{\mathcal{P}})$ is taken from Lemma 6.12.

Similarly, Lemma 6.12 shows that $(\gamma_j)_{j \in J}$ satisfies for $s := \min\{1, p_2\}$ the estimate

$$|\det S_j|^{s^{-1}-1} \cdot \left\| \mathcal{F}^{-1} \gamma_j \right\|_{L^s} \leq C = C(d, p_2, \varepsilon_{\mathcal{P}})$$

for all $j \in J$. Since we also have $\text{supp } \gamma_j \subset P_j$, we see that $\Gamma = (\gamma_j)_{j \in J}$ is an L^{p_2} -bounded system for \mathcal{P} , with $C_{\mathcal{P}, \Gamma, p_2} \leq C(d, p_2, \varepsilon_{\mathcal{P}})$ and $\ell_{\Gamma, \mathcal{P}} = 0$. By Theorem 3.17, this yields a positive constant $C_4 = C_4(d, p_2, \mathcal{P}, \varepsilon_{\mathcal{P}}, \|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z})$ with

$$\begin{aligned} \|\iota g^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)} &\geq C_4^{-1} \cdot \left\| \left(\left\| \mathcal{F}^{-1}(\gamma_j \cdot \iota g^{(r)}) \right\|_{L^{p_2}} \right)_{j \in J} \right\|_Z \\ &\geq C_4^{-1} \cdot \left\| \left(\left\| \mathcal{F}^{-1}(\gamma_j \cdot \iota g^{(r)}) \right\|_{L^{p_2}} \cdot \mathbf{1}_{J^{(r)}}(j) \right)_{j \in J} \right\|_Z. \end{aligned} \quad (6.77)$$

In particular, Theorem 3.17 shows that the sequence on the right-hand side of equation (6.77) is indeed an element of Z .

But we have $J^{(r)} = \bigsqcup_{i \in I^{(r)}} (J_0 \cap J_i)$ and $J_0 \cap J_i = \emptyset$ in case of $i \notin I_0$, i.e. $J^{(r)} = \bigsqcup_{i \in I_0 \cap I^{(r)}} (J_0 \cap J_i)$. Thus,

$$\begin{aligned} \|\mathcal{F}^{-1}(\gamma_j \cdot \iota g^{(r)})\|_{L^{p_2}} \cdot \mathbf{1}_{J^{(r)}}(j) &= \sum_{i \in I^{(r)} \cap I_0} \|\mathcal{F}^{-1}(\gamma_j \cdot \iota g^{(r)})\|_{L^{p_2}} \cdot \mathbf{1}_{J_0 \cap J_i}(j) \\ (\text{eq. (6.76)}) &= C_3 \cdot \sum_{i \in I^{(r)} \cap I_0} \zeta_i \cdot \left(|\det S_j|^{1-\frac{1}{p_2}} \cdot \mathbf{1}_{J_0 \cap J_i}(j) \right) \\ (d_i = 0 \text{ for } i \in I \setminus I_0) &= C_3 \cdot \sum_{i \in I^{(r)}} \left[d_i |\det T_i|^{\frac{1}{p_1}-1} \cdot \left(|\det S_j|^{1-\frac{1}{p_2}} \cdot \mathbf{1}_{J_0 \cap J_i}(j) \right) \right]. \end{aligned}$$

Now, summing over $r \in \underline{r_0}$ and using $I = \bigsqcup_{r=1}^{r_0} I^{(r)}$ show that the map η defined in the current theorem is well-defined. Furthermore, the (quasi)-triangle inequality for Z ensures existence of a constant $C_5 = C_5(r_0, C_Z) = C_5(N_{\mathcal{Q}}, k, C_Z)$ satisfying

$$\begin{aligned} \|\eta(d)\|_Z &\stackrel{(*)}{=} \left\| \sum_{i \in I} d_i |\det T_i|^{\frac{1}{p_1}-1} \cdot \left(|\det S_j|^{1-\frac{1}{p_2}} \cdot \mathbf{1}_{J_0 \cap J_i}(j) \right)_{j \in J} \right\|_Z \\ &\leq \sum_{r=1}^{r_0} \left\| \sum_{i \in I^{(r)}} d_i |\det T_i|^{\frac{1}{p_1}-1} \cdot \left(|\det S_j|^{1-\frac{1}{p_2}} \cdot \mathbf{1}_{J_0 \cap J_i}(j) \right)_{j \in J} \right\|_Z \\ &= C_3^{-1} \cdot \sum_{r=1}^{r_0} \left\| \left(\mathbf{1}_{J^{(r)}}(j) \cdot \|\mathcal{F}^{-1}(\gamma_j \cdot \iota g^{(r)})\|_{L^{p_2}} \right)_{j \in J} \right\|_Z \\ (\text{eq. (6.77)}) &\leq \frac{C_4}{C_3} \cdot \sum_{r=1}^{r_0} \|\iota g^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)} \\ &\leq \frac{C_4}{C_3} \|\iota\| \cdot \sum_{r=1}^{r_0} \|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, Y)} \\ &\leq r_0 \frac{C_1 C_2 C_4}{C_3} \|\iota\| \cdot \|d\|_{Y|_{I_0}}. \end{aligned}$$

Here, the first step (marked with $(*)$) used $d_i = 0$ for $i \in I \setminus I_0$. All in all, we have shown that η is well-defined and bounded, so that it remains to consider the second part of the lemma.

To this end, note that $J^{(r)} = \bigsqcup_{i \in I^{(r)}} (J_0 \cap J_i)$ implies

$$\|(c_j)_{j \in J^{(r)}}\|_{\ell^s} = \left\| \left(\|(c_j)_{j \in J_0 \cap J_i}\|_{\ell^s} \right)_{i \in I^{(r)}} \right\|_{\ell^s}$$

for arbitrary sequences $(c_j)_{j \in J^{(r)}}$ and any $s \in (0, \infty]$. If we apply this in equation (6.77), we get

$$\begin{aligned} \infty &> \|\iota g^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, Z)} \geq C_4^{-1} \cdot \left\| \left(v_j \cdot \|\mathcal{F}^{-1}(\gamma_j \cdot \iota g^{(r)})\|_{L^{p_2}} \right)_{j \in J^{(r)}} \right\|_{\ell^{q_2}} \\ &= C_4^{-1} \cdot \left\| \left(\left(v_j \cdot \|\mathcal{F}^{-1}(\gamma_j \cdot \iota g^{(r)})\|_{L^{p_2}} \right)_{j \in J_0 \cap J_i} \right)_{i \in I^{(r)}} \right\|_{\ell^{q_2}} \\ &\geq C_4^{-1} \cdot \left\| \left(\left(v_j \cdot \|\mathcal{F}^{-1}(\gamma_j \cdot \iota g^{(r)})\|_{L^{p_2}} \right)_{j \in J_0 \cap J_i} \right)_{i \in I^{(r)} \cap I_0} \right\|_{\ell^{q_2}} \\ (\text{eq. (6.76)}) &= \frac{C_3}{C_4} \cdot \left\| \left(\left(v_j \cdot \zeta_i \cdot |\det S_j|^{1-\frac{1}{p_2}} \right)_{j \in J_0 \cap J_i} \right)_{i \in I^{(r)} \cap I_0} \right\|_{\ell^{q_2}} \\ &= \frac{C_3}{C_4} \cdot \left\| \left(\left(d_i \cdot |\det T_i|^{\frac{1}{p_1}-1} \cdot \left(v_j \cdot |\det S_j|^{1-\frac{1}{p_2}} \right)_{j \in J_0 \cap J_i} \right)_{i \in I^{(r)} \cap I_0} \right) \right\|_{\ell^{q_2}} \\ &= \frac{C_3}{C_4} \cdot \|(d_i)_{i \in I^{(r)} \cap I_0}\|_{\ell_u^{q_2}(I_0 \cap I^{(r)})}. \end{aligned}$$

Now, as above, we sum over $r \in \underline{r_0}$ and use the (quasi)-triangle inequality for ℓ^{q_2} to obtain

$$\begin{aligned} \|d\|_{\ell_u^{q_2}(I_0)} &\leq C_5 \cdot \sum_{r=1}^{r_0} \left\| (d_i \cdot \mathbf{1}_{I^{(r)} \cap I_0}(i))_{i \in I_0} \right\|_{\ell_u^{q_2}(I_0)} \\ &\leq \frac{C_4 C_5}{C_3} \cdot \sum_{r=1}^{r_0} \left\| \iota g^{(r)} \right\|_{\mathcal{DF}(\mathcal{P}, L^{p_2}, Z)} \\ (\text{eq. (6.75)}) &\leq r_0 \frac{C_1 C_2 C_4 C_5}{C_3} \|\iota\| \cdot \|d\|_{Y|_{I_0}} < \infty \end{aligned}$$

for some constant $C_5 = C_5(r_0, q_2) = C_5(N_{\mathcal{Q}}, k, q_2)$. For $C' := r_0 \frac{C_1 C_2 C_4 C_5}{C_3}$, this shows $\|\gamma_0\| \leq C' \cdot \|\iota\|$ for $\gamma_0 : \ell_0(I_0) \cap Y|_{I_0} \hookrightarrow \ell_u^{q_2}(I_0)$.

Finally, since $\ell_u^{q_2}(I_0)$ satisfies the Fatou property, Lemma 4.6 yields boundedness of γ , as well as $\|\gamma\| \leq C' \cdot \|\iota\|$, as desired. Regarding the dependencies of the constants, note that Lemma 4.13 yields $\|\Gamma_{\mathcal{P}}\|_{\ell_v^{q_2} \rightarrow \ell_v^{q_2}} \leq C(q_2, N_{\mathcal{P}}, C_{v, \mathcal{P}})$. \square

Now, we specialize the above lemma to the case in which \mathcal{P}_{J_0} is relatively moderate with respect to \mathcal{Q} . It is worth noting that the imposed requirements are rather strict. In particular, for $J_0 = J$, they can never be fulfilled in case of $\mathcal{O} \cap \partial \mathcal{O}' \neq \emptyset$, as a consequence of Lemma 2.16 (with interchanged roles of \mathcal{Q}, \mathcal{P}). Furthermore, we remark that the following theorem is a slightly improved version of [23, Theorem 5.3.12] from my PhD thesis.

Theorem 6.21. Under the assumptions of Lemma 6.19, assume additionally that $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$ for certain $q_1, q_2 \in (0, \infty]$ and for certain weights $w = (w_i)_{i \in I}$ and $v = (v_j)_{j \in J}$ which are \mathcal{Q} -moderate and \mathcal{P} -moderate, respectively.

Furthermore, assume that \mathcal{Q} is tight and that

- (1) \mathcal{P}_{J_0} is relatively \mathcal{Q} -moderate.
- (2) The weight $v|_{J_0}$ is relatively \mathcal{Q} -moderate¹².
- (3) There is some $r \in \mathbb{N}_0$ and some $C_0 > 0$ such that we have

$$\lambda(Q_i) \leq C_0 \cdot \lambda \left(\bigcup_{j \in J_0 \cap J_i} P_j^{r*} \right)$$

for all $i \in I_0 := \{i \in I \mid J_0 \cap J_i \neq \emptyset\}$.

Let¹³ $s := \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\Delta}} \right)_+$ and choose for each $i \in I_0$ some $j_i \in J_0 \cap J_i$. Then

$$\left\| \left(\frac{v_{j_i}}{w_i} \cdot |\det T_i|^s \cdot |\det S_{j_i}|^{\frac{1}{p_1} - \frac{1}{p_2} - s} \right)_{i \in I_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \leq C \cdot \|\iota\|, \quad (6.78)$$

for some constant

$$C = C \left(d, r, C_0, p_1, p_2, q_1, q_2, \mathcal{Q}, \varepsilon_{\mathcal{Q}}, \mathcal{P}, \varepsilon_{\mathcal{P}}, k(\mathcal{P}_{J_0}, \mathcal{Q}), C_{\text{mod}}(\mathcal{P}_{J_0}, \mathcal{Q}), C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}, C_{v|_{J_0}, \mathcal{P}, \mathcal{Q}}, C_{\mathcal{Q}, \Phi, p_1} \right).$$

Here, the L^{p_1} -BAPU $\Phi = (\varphi_i)_{i \in I}$ has to be used to calculate the norm $\|\iota\|$. \blacktriangleleft

Remark. As in the remark after Lemma 2.17, we observe that assumption (3) is automatically satisfied (with $r = 0$ and $C_0 = 1$) if we have $\mathcal{O} = \mathcal{O}'$ and $J_0 = J$. \blacklozenge

Proof. Lemma 6.19 yields a constant $C_1 > 0$ which depends only on quantities mentioned in the current theorem¹⁴ and which satisfies $\|\gamma\| \leq C_1 \cdot \|\iota\|$, where

$$\gamma : \ell_w^{q_1}(I_0) \hookrightarrow \ell_u^{q_2}(I_0) \quad \text{with} \quad u_i := |\det T_i|^{\frac{1}{p_1} - 1} \cdot \left\| (v_j \cdot |\det S_j|^{1 - \frac{1}{p_2}})_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2}}.$$

¹²We recall that relative \mathcal{Q} -moderateness of $v|_{J_0}$ means that there is some $L = C_{v|_{J_0}, \mathcal{P}, \mathcal{Q}} > 0$ such that $v_j \leq L \cdot v_\ell$ holds for all $j, \ell \in J_0 \cap J_i$ for arbitrary $i \in I$.

¹³Recall from Remark 5.13 that $p^{\pm\Delta} \in \mathbb{R} \cup \{\infty\}$ is defined by $1/p^{\pm\Delta} = \min \left\{ p, 1 - \frac{1}{p} \right\}$.

¹⁴This uses Lemma 4.13 to estimate $\|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}$ and $\|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}$ for $Y = \ell_w^{q_1}(I)$ and $Z = \ell_v^{q_2}(J)$ in terms of $C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}, N_{\mathcal{Q}}, N_{\mathcal{P}}$ and q_1, q_2 . Furthermore, it is used that C_Z only depends on q_2 .

In view of Lemma 4.8, this yields

$$\left\| \left(\frac{|\det T_i|^{\frac{1}{p_1}-1}}{w_i} \cdot \left\| \left(v_j \cdot |\det S_j|^{1-\frac{1}{p_2}} \right)_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2}} \right)_{i \in I_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} = \|(u_i/w_i)_{i \in I_0}\|_{\ell^{q_2 \cdot (q_1/q_2)'}} = \|\gamma\| \leq C_1 \cdot \|\iota\|. \quad (6.79)$$

Next, set $C_2 := C_{\text{mod}}(\mathcal{P}_{J_0}, \mathcal{Q})$ and $L := C_{v|_{J_0}, \mathcal{P}, \mathcal{Q}} > 0$, so that $v_j \leq L \cdot v_\ell$ holds for all $j, \ell \in J_0 \cap J_i$ for arbitrary $i \in I$. In particular, this implies $v_j \geq L^{-1} \cdot v_{j_i}$ for all $j \in J_0 \cap J_i$ and arbitrary $i \in I_0$. In addition, by choice of C_2 , we have

$$C_2^{-1} \cdot |\det S_j| \leq |\det S_{j_i}| \leq C_2 \cdot |\det S_j| \quad \forall i \in I_0 \text{ and } j \in J_0 \cap J_i, \quad (6.80)$$

since $j_i \in J_0 \cap J_i$ as well. But this yields a constant $C_3 = C_3(C_{\text{mod}}(\mathcal{P}_{J_0}, \mathcal{Q}), p_2) \geq 1$ with

$$|\det S_j|^{1-\frac{1}{p_2}} \geq C_3^{-1} \cdot |\det S_{j_i}|^{1-\frac{1}{p_2}} \quad \forall i \in I_0 \text{ and } j \in J_0 \cap J_i. \quad (6.81)$$

Note that this remains true also in case of $1 - \frac{1}{p_2} < 0$.

Next, note that the assumptions of the present theorem include the prerequisites of Lemma 2.17 (with interchanged roles of \mathcal{Q} and \mathcal{P}), so that there is a constant

$$C_4 = C_4(C_0, d, r, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{Q}}, C_{\text{mod}}(\mathcal{P}_{J_0}, \mathcal{Q})) > 0$$

with

$$|\det T_i| / |\det S_{j_i}| \leq C_4 \cdot |J_0 \cap J_i| \quad (6.82)$$

for all $i \in I_0$. Together with equation (6.80), this implies

$$\begin{aligned} \left\| \left(v_j \cdot |\det S_j|^{1-\frac{1}{p_2}} \right)_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2}} &\geq C_3^{-1} L^{-1} \cdot |\det S_{j_i}|^{1-\frac{1}{p_2}} v_{j_i} \cdot |J_0 \cap J_i|^{1/q_2} \\ &\geq \left(C_3 C_4^{1/q_2} L \right)^{-1} \cdot |\det S_{j_i}|^{1-\frac{1}{p_2}-\frac{1}{q_2}} |\det T_i|^{\frac{1}{q_2}} \cdot v_{j_i} \end{aligned}$$

for all $i \in I_0$.

Let us set $C_5 := C_1 C_3 C_4^{1/q_2} L$. By combining the estimate above with equation (6.79), we conclude

$$\left\| \left(\frac{v_{j_i}}{w_i} \cdot |\det T_i|^{\frac{1}{q_2} + \frac{1}{p_1} - 1} |\det S_{j_i}|^{1-\frac{1}{p_2}-\frac{1}{q_2}} \right)_{i \in I_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \leq C_5 \|\iota\|. \quad (6.83)$$

With this preparation, we can deduce the actual claim of the theorem. For technical reasons, however, we need to distinguish three cases:

Case 1: $p_1 \in (0, 1)$. In this case, we have

$$\frac{1}{p_1^{\pm \Delta}} = \min \left\{ \frac{1}{p_1}, 1 - \frac{1}{p_1} \right\} = 1 - \frac{1}{p_1} < 0$$

and hence

$$s = \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm \Delta}} \right)_+ = \frac{1}{q_2} + \frac{1}{p_1} - 1.$$

Thus,

$$|\det T_i|^s \cdot |\det S_{j_i}|^{\frac{1}{p_1} - \frac{1}{p_2} - s} = |\det T_i|^{\frac{1}{q_2} + \frac{1}{p_1} - 1} \cdot |\det S_{j_i}|^{1 - \frac{1}{p_2} - \frac{1}{q_2}},$$

so that the desired estimate (6.78) is a direct consequence of equation (6.83).

Case 2: $p_1 \in [1, 2]$ and $q_2 \leq p'_1$. Here, we have

$$\frac{1}{p_1^{\pm \Delta}} = \min \left\{ \frac{1}{p_1}, 1 - \frac{1}{p_1} \right\} = 1 - \frac{1}{p_1} = \frac{1}{p'_1}$$

and hence $\frac{1}{q_2} - \frac{1}{p_1^{\pm \Delta}} = \frac{1}{q_2} - \frac{1}{p'_1} \geq 0$, which yields

$$s = \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm \Delta}} \right)_+ = \frac{1}{q_2} - \frac{1}{p'_1} = \frac{1}{q_2} + \frac{1}{p_1} - 1.$$

Thus, we see exactly as in the previous case that the desired estimate is a direct consequence of equation (6.83).

Case 3: We have $p_1 \in [1, 2]$ and $q_2 > p'_1$ or we have $p_1 \in [2, \infty]$. In this case, we will *not* use equation (6.83). Instead, we will invoke Theorem 6.13. As a preparation, set $t := q_2 \cdot (p_1/q_2)'$ and note that the exponent $s = \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\Delta}}\right)_+$ satisfies

$$s = \frac{1}{q_2 \cdot (p_1/q_2)'} = \frac{1}{t}. \quad (6.84)$$

To see this, we first note that equation (4.2) shows $\frac{1}{t} = \left(\frac{1}{q_2} - \frac{1}{p_1}\right)_+$. To complete the proof of equation (6.84), we distinguish two sub-cases:

Case 1. We have $p_1 \in [1, 2]$ and $q_2 > p'_1$. In this case, note $p_1^{\pm\Delta} = p_1^\Delta = \max\{p_1, p'_1\} = p'_1$ and hence $\frac{1}{q_2} - \frac{1}{p_1^{\pm\Delta}} = \frac{1}{q_2} - \frac{1}{p'_1} < 0$, since $q_2 > p'_1$. This implies $s = 0$. But since $p_1 \in [1, 2]$, we also have $p_1 \leq p'_1$ and thus

$$\frac{1}{q_2} - \frac{1}{p_1} \leq \frac{1}{q_2} - \frac{1}{p'_1} < 0,$$

so that $s = 0 = \left(\frac{1}{q_2} - \frac{1}{p_1}\right)_+ = \frac{1}{t}$, as desired.

Case 2. We have $p_1 \in [2, \infty]$. In this case, note $p_1^{\pm\Delta} = p_1^\Delta = \max\{p_1, p'_1\} = p_1$, which entails $s = \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\Delta}}\right)_+ = \left(\frac{1}{q_2} - \frac{1}{p_1}\right)_+ = \frac{1}{t}$, as claimed.

Recall from Lemma 6.19 that $k = k(\mathcal{P}_{J_0}, \mathcal{Q})$. Now, to show that Theorem 6.13 is applicable, note for $j \in J_0$ that we have $\emptyset \neq P_j \subset Q_{i_j}^{k*} \subset \mathcal{O}$ for some $i_j \in I$. In particular, $P_j \cap Q_i \neq \emptyset$ for some $i \in I$, which implies $i \in I_0$ and (by Lemma 2.11) that

$$P_j \subset Q_i^{(2k+2)*} \subset \overline{Q_i^{(2k+3)*}}.$$

Thus, using the notation of Lemma 6.19, we have

$$K = \bigcup_{i \in I_0} \overline{Q_i^{(2k+3)*}} \supset \bigcup_{j \in J_0} P_j.$$

In view of the assumed boundedness of ι (cf. Lemma 6.19), we thus see that the prerequisites of Theorem 6.13 are satisfied (with $\|\iota_0\| \leq \|\iota\|$, where ι_0 denotes the embedding from Theorem 6.13). Since $Z = \ell_v^{q_2}(J)$ satisfies the Fatou property, since the triangle inequality of Z only depends on q_2 and since Lemma 4.13 shows $\|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z} \leq C_{v, \mathcal{P}} \cdot N_{\mathcal{P}}^{1+\frac{1}{q_2}}$, we conclude that there is a constant

$$C_6 = C_6(d, k(\mathcal{P}_{J_0}, \mathcal{Q}), p_1, p_2, q_2, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{P}}, C_{v, \mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1})$$

satisfying $\|\eta\| \leq C_6 \cdot \|\iota\|$, for

$$\eta : \ell_w^{q_1}([\ell^{p_1}(J_i \cap J_0)]_{i \in I}) \hookrightarrow \ell_{v_j \cdot |\det S_j|^{p_1^{-1} - p_2^{-1}}}^{q_2}(J_0).$$

Now, an application of Corollary 5.12 (with $r = p_1$, $u^{(1)} \equiv 1$ and $u^{(2)} \equiv 1$ and finally with $(v_j \cdot |\det S_j|^{p_1^{-1} - p_2^{-1}})_{j \in J}$ instead of v) yields a constant $C_7 = C_7(p_1, q_1, q_2, \mathcal{Q}, C_{w, \mathcal{Q}}, k(\mathcal{P}_{J_0}, \mathcal{Q}))$ such that

$$\left\| \left(w_i^{-1} \cdot \left(v_j \cdot |\det S_j|^{p_1^{-1} - p_2^{-1}} \right)_{j \in J_0 \cap J_i} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (p_1/q_2)'}(I)} \leq C_7 \|\eta\| \leq C_6 C_7 \|\iota\|.$$

Now, the same arguments as in the previous cases—using relative \mathcal{Q} -moderateness of \mathcal{P}_{J_0} and of $v|_{J_0}$ —see in particular equation (6.82)—show

$$\begin{aligned} v_{j_i} |\det S_{j_i}|^{p_1^{-1}-p_2^{-1}-s} \cdot |\det T_i|^s &\leq C_4^s \cdot v_{j_i} |\det S_{j_i}|^{p_1^{-1}-p_2^{-1}} \cdot |J_0 \cap J_i|^s \\ &\left(\text{since } s = \frac{1}{t} \right) = C_4^{1/t} \cdot v_{j_i} |\det S_{j_i}|^{p_1^{-1}-p_2^{-1}} \cdot |J_0 \cap J_i|^{\frac{1}{t}} \\ &\leq C_4^{1/t} C_8 \cdot \left\| \left(v_j \cdot |\det S_j|^{p_1^{-1}-p_2^{-1}} \right)_{j \in J_0 \cap J_i} \right\|_{\ell^t} \\ &= C_4^{1/t} C_8 \cdot \left\| \left(v_j \cdot |\det S_j|^{p_1^{-1}-p_2^{-1}} \right)_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2 \cdot (p_1/q_2)'}} \end{aligned}$$

for some constant $C_8 = C_8(p_1, p_2, C_{v|_{J_0, \mathcal{P}, \mathcal{Q}}}, C_{\text{mod}}(\mathcal{P}_{J_0}, \mathcal{Q})) = C_8(p_1, p_2, C_2, L)$ and all $i \in I_0$.

All in all, we conclude

$$\begin{aligned} &\left\| \left(\frac{v_{j_i}}{w_i} \cdot |\det S_{j_i}|^{p_1^{-1}-p_2^{-1}-s} \cdot |\det T_i|^s \right)_{i \in I_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ &\leq C_4^{1/t} C_8 \cdot \left\| \left(w_i^{-1} \cdot \left\| \left(v_j \cdot |\det S_j|^{p_1^{-1}-p_2^{-1}} \right)_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2 \cdot (p_1/q_2)'}} \right)_{i \in I_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ &\leq C_4^{1/t} C_8 \cdot \left\| \left(w_i^{-1} \cdot \left\| \left(v_j \cdot |\det S_j|^{p_1^{-1}-p_2^{-1}} \right)_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2 \cdot (p_1/q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ &\leq C_4^{1/t} C_6 C_7 C_8 \cdot \| \iota \|, \end{aligned}$$

as desired. \square

With similar techniques, we will now prove the analogous result for the “reverse” case in which (a subfamily of) \mathcal{Q} is almost subordinate to and relatively moderate with respect to \mathcal{P} . But as above, we first establish an auxiliary result for which we do *not* need to assume that \mathcal{Q} is relatively moderate with respect to \mathcal{P} . In contrast to Lemma 6.19, however, we assume the “global” components Y, Z to be weighted Lebesgue spaces, since the proof will use (a slight variant of) the “reproducing” property $\ell_w^{q_1}(I) = \ell^{q_1}([\ell_w^{q_1}(I^{(k)})]_{k \in K})$, which holds for arbitrary partitions $I = \bigsqcup_{k \in K} I^{(k)}$.

Lemma 6.22. *Let $\emptyset \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ be a semi-structured L^{p_1} -decomposition covering of \mathcal{O} and let $\mathcal{P} = (P_j)_{j \in J} = (S_j P'_j + c_j)_{j \in J}$ be a semi-structured L^{p_2} -decomposition covering of \mathcal{O}' , for certain $p_1, p_2 \in (0, \infty]$. Let $w = (w_i)_{i \in I}$ and $v = (v_j)_{j \in J}$ be \mathcal{Q} -moderate and \mathcal{P} -moderate, respectively. Finally, let $q_1, q_2 \in (0, \infty]$.*

Choose an arbitrary subset $J_0 \subset J$ such that there is¹⁵ some $\varepsilon > 0$ and for every $j \in J_0$ some $\xi_j \in \mathbb{R}^d$ with $B_\varepsilon(\xi_j) \subset P'_j$ and $S_j[B_\varepsilon(\xi_j)] + c_j \subset \mathcal{O}$.

Define

$$I_0 := \{i \in I \mid J_i \cap J_0 \neq \emptyset\}$$

and assume that $\mathcal{Q}_{I_0} := (Q_i)_{i \in I_0}$ is almost subordinate to \mathcal{P} and that¹⁶

$$u_j := \left\| \left(w_i \cdot |\det T_i|^{1-\frac{1}{p_1}} \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} = \left\| \left(w_i \cdot |\det T_i|^{1-\frac{1}{p_1}} \right)_{i \in I_j} \right\|_{\ell^{q_1}} < \infty \quad (6.85)$$

for all $j \in J_0$.

Finally, set

$$K := \bigcup_{i \in I_0} Q_i \subset \mathcal{O} \cap \mathcal{O}'$$

¹⁵The easiest case in which this assumption is fulfilled is if \mathcal{P} is tight with $P_j \subset \mathcal{O}$ for all $j \in J_0$. In this case, one can simply take $\varepsilon = \varepsilon_{\mathcal{P}}$.

¹⁶Note that each $j \in J_0$ satisfies $\emptyset \neq S_j[B_\varepsilon(\xi_j)] + c_j \subset P_j \cap \mathcal{O}$ by assumption. Hence, there is some $i \in I$ with $Q_i \cap P_j \neq \emptyset$, which yields $i \in I_0 \cap I_j \neq \emptyset$. In particular, the quantity in equation (6.85) is always positive. Finally, the equality in equation (6.85) is justified by equation (6.88) below, since this equation yields $I_j \subset I_0$ and hence $I_0 \cap I_j = I_j$ for all $j \in J_0$.

and assume that the map

$$\iota : \left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})} \right) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2}), f \mapsto f$$

is bounded¹⁷. Then we have

$$\left\| \left(v_j \cdot |\det S_j|^{1-\frac{1}{p_2}} / u_j \right)_{j \in J_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'} } \leq C \|\iota\|$$

for some constant

$$C = C(d, p_1, p_2, q_2, \varepsilon, k(\mathcal{Q}_{I_0}, \mathcal{P}), \mathcal{Q}, \mathcal{P}, C_{v, \mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}).$$

Here, the L^{p_1}/L^{p_2} BAPUs $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_j)_{j \in J}$ are those which are used to calculate the norm $\|\iota\|$. \blacktriangleleft

Proof. For brevity, set $k := k(\mathcal{Q}_{I_0}, \mathcal{P})$.

Using the inclusion $S_j[B_\varepsilon(\xi_j)] + c_j \subset P_j \cap \mathcal{O}$, one can use exactly the same construction as in the proof of Lemma 6.12 (take a nontrivial $\gamma \in C_c^\infty(B_1(0))$ and define $\gamma_j := L_{c_j}(L_{\xi_j}[\gamma(\varepsilon^{-1} \cdot)] \circ S_j^{-1})$ for $j \in J_0$) to obtain a family $(\gamma_j)_{j \in J_0}$ satisfying $\gamma_j \in C_c^\infty(P_j \cap \mathcal{O}) \subset C_c^\infty(\mathcal{O} \cap \mathcal{O}')$ and

$$\|\mathcal{F}^{-1}\gamma_j\|_{L^p} = C_1^{(p)} \cdot |\det S_j|^{1-\frac{1}{p}} \quad (6.86)$$

for all $j \in J_0$, $p \in (0, \infty]$ and a constant $C_1^{(p)} = C_1^{(p)}(d, \varepsilon) > 0$.

We start with a few technical observations which will be useful later on. We first note

$$K' := \bigcup_{j \in J_0} (P_j \cap \mathcal{O}) \subset K. \quad (6.87)$$

To see this, let $\xi \in K'$ be arbitrary. Hence, $\xi \in P_j \cap \mathcal{O}$ for some $j \in J_0$ and thus $\xi \in Q_i$ for some $i \in I$, since \mathcal{Q} covers \mathcal{O} . This entails $\xi \in Q_i \cap P_j \neq \emptyset$ and hence $j \in J_i \cap J_0 \neq \emptyset$, which implies $i \in I_0$ and thus finally $\xi \in Q_i \subset \bigcup_{\ell \in I_0} Q_\ell = K$.

Furthermore, we have

$$I_j \subset I_0 \text{ for all } j \in J_0, \quad (6.88)$$

because for arbitrary $i \in I_j$ we have $j \in J_i \cap J_0 \neq \emptyset$ and hence $i \in I_0$.

Finally, we note

$$|J_i| \leq N_{\mathcal{P}}^{k+1} \text{ for all } i \in I_0. \quad (6.89)$$

To see that this is true, note that $j \in J_i$ entails $\emptyset \neq Q_i \cap P_j \subset P_{j_i}^{k*} \cap P_j$ for some $j_i \in J$ and thus $j \in j_i^{(k+1)*}$, which means $J_i \subset j_i^{(k+1)*}$. Using Lemma 2.9, this easily implies that equation (6.89) is true.

Now, set $r_0 := N_{\mathcal{P}}^{2 \cdot 1+1} = N_{\mathcal{P}}^3$, so that Lemma 2.14 ensures existence of a partition $J = \bigsqcup_{r=1}^{r_0} J^{(r)}$ with $P_j^* \cap P_m^* = \emptyset$ for all $j, m \in J^{(r)}$ with $j \neq m$ and arbitrary $r \in \underline{r_0}$.

Let $d = (d_j)_{j \in J_0} \in \ell_0(J_0)$ be arbitrary. Fix $r \in \underline{r_0}$ and define

$$g^{(r)} := \sum_{j \in J_0 \cap J^{(r)}} d_j \gamma_j.$$

Equation (6.87) and compactness of $\text{supp } \gamma_j \subset P_j \cap \mathcal{O} \subset K'$ show that $\text{supp } g^{(r)} \subset K \subset \mathcal{O} \cap \mathcal{O}'$ is compact, since the sum defining $g^{(r)}$ is finite. This easily yields $g^{(r)} \in \mathcal{D}_K \leq \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$. As usual, we now derive an upper bound on $\|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})}$ and a lower bound on $\|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})}$.

Let us begin with the lower bound: For $j \in J_0 \cap J^{(r)}$ and $m \in J^{(r)}$ with $\psi_m^* \gamma_j \neq 0$, we have

$$\emptyset \neq P_m^* \cap \text{supp } \gamma_j \subset P_m^* \cap P_j \subset P_m^* \cap P_j^*$$

¹⁷Here, $\mathcal{D}_K := \{f \in C^\infty(\mathbb{R}^d) \mid \text{supp } f \subset K\} \leq C_c^\infty(\mathcal{O} \cap \mathcal{O}')$. Note that Lemma 2.4 shows that $(\varphi_i)_{i \in I}$ is a locally finite partition of unity on \mathcal{O} . Hence, for each $g \in \mathcal{D}_K$, we can have $\varphi_i g \neq 0$ only for finitely many $i \in I$. Since $\ell_w^{q_1}(I)$ contains all finitely supported sequences, we easily see $\mathcal{D}_K \leq \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$. An analogous argument shows $\mathcal{D}_K \leq \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$, so that ι is well-defined—but not necessarily bounded.

and thus $m = j$ by choice of $J^{(r)}$. But Lemma 2.4 shows $\psi_m^* \equiv 1$ on $P_m = P_j \supset \text{supp } \gamma_j$ and thus $\psi_m^* \gamma_j = \gamma_j = \gamma_m$. In summary, we conclude

$$\psi_m^* g^{(r)} = d_m \cdot \gamma_m \quad \forall m \in J_0 \cap J^{(r)}. \quad (6.90)$$

Next, note that Remark 3.16 shows that $\Gamma = (\psi_m^*)_{m \in J}$ is an L^{p_2} -bounded (control) system for \mathcal{P} with $C_{\mathcal{P}, \Gamma, p_2} \leq C_2 = C_2(d, p_2, \mathcal{P}, C_{\mathcal{P}, \Psi, p_2})$. In combination with Theorem 3.17 and with the estimate $\|\Gamma \mathcal{P}\|_{\ell_v^{q_2} \rightarrow \ell_v^{q_2}} \leq C_{v, \mathcal{P}} \cdot N_{\mathcal{P}}^{1+q_2^{-1}}$ from Lemma 4.13, we get

$$\begin{aligned} C_2 C_3 \cdot \|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})} &\geq \left\| \left(\left\| \mathcal{F}^{-1}(\psi_m^* g^{(r)}) \right\|_{L^{p_2}} \right)_{m \in J} \right\|_{\ell_v^{q_2}} \\ &\geq \left\| \left(\left\| \mathcal{F}^{-1}(\psi_m^* g^{(r)}) \right\|_{L^{p_2}} \right)_{m \in J_0 \cap J^{(r)}} \right\|_{\ell_v^{q_2}} \\ (\text{eq. (6.90)}) &= \left\| \left(\left\| \mathcal{F}^{-1}(d_m \cdot \gamma_m) \right\|_{L^{p_2}} \right)_{m \in J_0 \cap J^{(r)}} \right\|_{\ell_v^{q_2}} \\ (\text{eq. (6.86)}) &= C_1^{(p_2)} \cdot \left\| \left(|\det S_m|^{1-\frac{1}{p_2}} \cdot d_m \right)_{m \in J_0 \cap J^{(r)}} \right\|_{\ell_v^{q_2}} \end{aligned} \quad (6.91)$$

for some constant $C_3 = C_3(d, q_2, p_2, \mathcal{P}, C_{v, \mathcal{P}}) > 0$.

Next, we want to establish an upper bound for $\|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})}$. To this end, consider any $i \in I$ with

$$0 \neq \varphi_i \cdot g^{(r)} = \sum_{j \in J_0 \cap J^{(r)}} [d_j \cdot \varphi_i \gamma_j].$$

Then, there is some $j \in J_0 \cap J^{(r)}$ satisfying $0 \neq \varphi_i \gamma_j$. But for *any*(!) such $j \in J_0 \cap J^{(r)}$, we get $Q_i \cap P_j \neq \emptyset$, which leads to $i \in I_j \subset I_0$ (cf. equation (6.88)), or equivalently $i \in I_0$ and $j \in J_i$. Hence,

$$\varphi_i g^{(r)} = \sum_{j \in J_0 \cap J^{(r)} \cap J_i} [d_j \cdot \varphi_i \gamma_j] \quad (6.92)$$

and $i \in I_0$.

Now, we distinguish the cases $p_1 \in [1, \infty]$ and $p_1 \in (0, 1)$. For $p_1 \in [1, \infty]$, we use equation (6.92), Young's inequality ($L^1 * L^{p_1} \hookrightarrow L^{p_1}$) and the triangle inequality for $\|\cdot\|_{L^{p_1}}$, to derive

$$\begin{aligned} w_i \cdot \|\mathcal{F}^{-1}(\varphi_i g^{(r)})\|_{L^{p_1}} &= w_i \cdot \left\| \sum_{j \in J_0 \cap J_i \cap J^{(r)}} \mathcal{F}^{-1}(d_j \cdot \varphi_i \gamma_j) \right\|_{L^{p_1}} \\ &\leq w_i \cdot \sum_{j \in J_0 \cap J_i \cap J^{(r)}} [|d_j| \cdot \|\mathcal{F}^{-1} \gamma_j\|_{L^1} \|\mathcal{F}^{-1} \varphi_i\|_{L^{p_1}}] \\ (\text{equations (6.86) and (6.32)}) &\leq C_1^{(1)} C_4 \cdot |\det T_i|^{1-\frac{1}{p_1}} w_i \cdot \sum_{j \in J_0 \cap J_i \cap J^{(r)}} |d_j|, \end{aligned}$$

where the left-hand side of the estimate vanishes for $i \in I \setminus I_0$, as the discussion above showed. In the previous estimate, the last step used that Lemma 6.12 yields

$$\|\mathcal{F}^{-1} \varphi_i\|_{L^{p_1}} \leq C_4 \cdot |\det T_i|^{1-\frac{1}{p_1}}, \quad (6.93)$$

for all $i \in I$ and some constant $C_4 = C_4(d, p_1, \mathcal{Q}, C_{\mathcal{Q}, \Phi, p_1})$.

In case of $p_1 \in (0, 1)$, we note that $\|\cdot\|_{L^{p_1}}$ is a quasi-norm with triangle constant only depending on p_1 . Together with the uniform bound $|J_0 \cap J_i \cap J^{(r)}| \leq |J_i| \leq N_{\mathcal{P}}^{k+1}$ from equation (6.89)—and recalling the identity (6.92)—this yields a constant $C_5 = C_5(p_1, k, N_{\mathcal{P}})$ satisfying

$$\begin{aligned} w_i \cdot \|\mathcal{F}^{-1}(\varphi_i g^{(r)})\|_{L^{p_1}} &= w_i \cdot \left\| \sum_{j \in J_0 \cap J_i \cap J^{(r)}} \mathcal{F}^{-1}(d_j \cdot \varphi_i \gamma_j) \right\|_{L^{p_1}} \\ &\leq C_5 \cdot w_i \cdot \sum_{j \in J_0 \cap J_i \cap J^{(r)}} |d_j| \|\mathcal{F}^{-1}(\varphi_i \gamma_j)\|_{L^{p_1}}. \end{aligned}$$

Now, observe for $i \in I_0$ and $j \in J_0 \cap J_i \cap J^{(r)}$ that we have $\text{supp } \varphi_i \subset \overline{Q_i} \subset \overline{P_j^{(2k+2)^*}}$ as well as $\text{supp } \gamma_j \subset P_j \subset \overline{P_j^{(2k+2)^*}}$. Indeed, $Q_i \subset P_j^{(2k+2)^*}$ is a consequence of $Q_i \cap P_j \neq \emptyset$ (since $j \in J_i$) and of Lemma 2.11, together with the fact that \mathcal{Q}_{I_0} is almost subordinate to \mathcal{P} with $k = k(\mathcal{Q}_{I_0}, \mathcal{P})$. Thus, Corollary 3.14 yields a constant $C_6 = C_6(d, p_1, k, \mathcal{P}) > 0$ with

$$\begin{aligned} \|\mathcal{F}^{-1}(\varphi_i \gamma_j)\|_{L^{p_1}} &\leq C_6 \cdot |\det S_j|^{\frac{1}{p_1}-1} \cdot \|\mathcal{F}^{-1}\varphi_i\|_{L^{p_1}} \cdot \|\mathcal{F}^{-1}\gamma_j\|_{L^{p_1}} \\ (\text{eq. (6.86)}) &= C_1^{(p_1)} C_6 \cdot \|\mathcal{F}^{-1}\varphi_i\|_{L^{p_1}} \\ (\text{def. of an } L^{p_1}\text{-BAPU}) &\leq C_1^{(p_1)} C_6 C_{\mathcal{Q}, \Phi, p_1} \cdot |\det T_i|^{1-\frac{1}{p_1}}. \end{aligned}$$

All in all, if we set $C_7 := C_1^{(p_1)} C_5 C_6 C_{\mathcal{Q}, \Phi, p_1}$, we arrive at

$$w_i \cdot \|\mathcal{F}^{-1}(\varphi_i g^{(r)})\|_{L^{p_1}} \leq C_7 \cdot |\det T_i|^{1-\frac{1}{p_1}} w_i \cdot \sum_{j \in J_0 \cap J_i \cap J^{(r)}} |d_j|, \quad (6.94)$$

where the left-hand side vanishes for $i \notin I_0$. Together with the case $p_1 \in [1, \infty]$ considered above, we conclude that this estimate holds for all $p_1 \in (0, \infty]$, with $C_7 := C_1^{(1)} C_4$ in case of $p_1 \in [1, \infty]$.

Now, we can estimate $\|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})}$. We first assume $q_1 < \infty$ and make use of equation (6.89), which yields $|J_0 \cap J_i \cap J^{(r)}| \leq N_{\mathcal{P}}^{k+1}$, to derive

$$\left(\sum_{j \in J_0 \cap J_i \cap J^{(r)}} \theta_j \right)^{q_1} \leq \left(|J_0 \cap J_i \cap J^{(r)}| \cdot \max_{j \in J_0 \cap J_i \cap J^{(r)}} \theta_j \right)^{q_1} \leq (N_{\mathcal{P}}^{k+1})^{q_1} \cdot \sum_{j \in J_0 \cap J_i \cap J^{(r)}} \theta_j^{q_1},$$

for arbitrary non-negative sequences $(\theta_j)_j$. We then use equation (6.94) and set $C_8 := C_7 N_{\mathcal{P}}^{k+1}$, to deduce

$$\begin{aligned} \|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})} &= \left\| \left(\|\mathcal{F}^{-1}(\varphi_i g^{(r)})\|_{L^{p_1}} \right)_{i \in I} \right\|_{\ell_w^{q_1}} \\ (\text{l.h.s. of (6.94) vanishes for } i \notin I_0) &\leq C_8 \cdot \left(\sum_{i \in I_0} \left[\left(|\det T_i|^{1-\frac{1}{p_1}} w_i \right)^{q_1} \sum_{j \in J_0 \cap J_i \cap J^{(r)}} |d_j|^{q_1} \right] \right)^{\frac{1}{q_1}} \\ &= C_8 \cdot \left(\sum_{j \in J_0 \cap J^{(r)}} \left[|d_j|^{q_1} \sum_{\substack{i \in I_0 \\ \text{with } j \in J_i}} \left(|\det T_i|^{1-\frac{1}{p_1}} w_i \right)^{q_1} \right] \right)^{\frac{1}{q_1}} \\ (j \in J_i \iff i \in I_j) &\leq C_8 \cdot \left\| \left(d_j \cdot \left\| \left(|\det T_i|^{1-\frac{1}{p_1}} w_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} \right)_{j \in J_0} \right\|_{\ell^{q_1}}. \quad (6.95) \end{aligned}$$

Now, let us consider the case $q_1 = \infty$. Here, we have

$$\begin{aligned} \|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})} &= \sup_{i \in I} [w_i \cdot \|\mathcal{F}^{-1}(\varphi_i g^{(r)})\|_{L^{p_1}}] \\ (\text{l.h.s. of (6.94) vanishes for } i \notin I_0) &\leq C_7 \cdot \sup_{i \in I_0} \left[|\det T_i|^{1-\frac{1}{p_1}} w_i \cdot \sum_{j \in J_0 \cap J_i \cap J^{(r)}} |d_j| \right] \\ (\text{since } |J_0 \cap J_i \cap J^{(r)}| \leq N_{\mathcal{P}}^{k+1}) &\leq C_7 N_{\mathcal{P}}^{k+1} \cdot \sup_{i \in I_0} \left[|\det T_i|^{1-\frac{1}{p_1}} w_i \cdot \sup_{j \in J_0 \cap J_i \cap J^{(r)}} |d_j| \right] \\ &= C_7 N_{\mathcal{P}}^{k+1} \cdot \sup_{j \in J_0 \cap J^{(r)}} \left[|d_j| \cdot \sup_{\substack{i \in I_0 \\ \text{with } j \in J_i}} |\det T_i|^{1-\frac{1}{p_1}} w_i \right] \\ (j \in J_i \iff i \in I_j \text{ and } q_1 = \infty) &\leq C_8 \cdot \left\| \left(d_j \cdot \left\| \left(|\det T_i|^{1-\frac{1}{p_1}} w_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} \right)_{j \in J_0} \right\|_{\ell^{q_1}}. \quad (6.96) \end{aligned}$$

Observe that the right-hand sides of equations (6.95) and (6.96) are finite, since $(d_j)_{j \in J_0} \in \ell_0(J_0)$ and because of equation (6.85).

All in all, using the weight u_j defined in equation (6.85), we see that equations (6.95) and (6.96) show

$$\|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})} \leq C_8 \cdot \|(u_j \cdot d_j)_{j \in J_0}\|_{\ell_{q_1}} \quad (6.97)$$

for arbitrary $q_1 \in (0, \infty]$.

Now, we finish the proof: We use estimates (6.91) and (6.97), as well as boundedness of ι , to deduce

$$\begin{aligned} & \left\| \left(d_m \cdot |\det S_m|^{1-\frac{1}{p_2}} \right)_{m \in J_0 \cap J^{(r)}} \right\|_{\ell_v^{q_2}} \\ (\text{eq. (6.91)}) & \stackrel{(\dagger)}{\leq} \frac{C_2 C_3}{C_1^{(p_2)}} \cdot \|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Psi}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})} \\ & \leq \frac{C_2 C_3}{C_1^{(p_2)}} \cdot \|\iota\| \cdot \|g^{(r)}\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})} \\ (\text{eq. (6.97)}) & \leq \frac{C_2 C_3 C_8}{C_1^{(p_2)}} \cdot \|\iota\| \cdot \|(u_j \cdot d_j)_{j \in J_0}\|_{\ell_{q_1}} \end{aligned} \quad (6.98)$$

for all sequences $d = (d_j)_{j \in J_0} \in \ell_0(J_0)$.

Since $\ell_v^{q_2}(J_0)$ is a quasi-normed space with triangle constant only depending on q_2 , and because of $J_0 = \biguplus_{r=1}^{r_0} (J^{(r)} \cap J_0)$, there is a constant $C_9 = C_9(q_2, r_0) > 0$ with

$$\|(d_m)_{m \in J_0}\|_{\ell_v^{q_2}} \leq C_9 \cdot \sum_{r=1}^{r_0} \|(d_m)_{m \in J^{(r)} \cap J_0}\|_{\ell_v^{q_2}}$$

for arbitrary sequences $(d_j)_{j \in J_0}$. Combining this with estimate (6.98) implies

$$\left\| \left(d_j \cdot |\det S_j|^{1-\frac{1}{p_2}} \right)_{j \in J_0} \right\|_{\ell_v^{q_2}} \leq \frac{C_2 C_3 C_8 C_9 r_0}{C_1^{(p_2)}} \cdot \|\iota\| \cdot \|(u_j \cdot d_j)_{j \in J_0}\|_{\ell_{q_1}}$$

for arbitrary sequences $(d_j)_{j \in J_0} \in \ell_0(J_0)$, which simply means that the embedding

$$\eta : \ell_0(J_0) \cap \ell_u^{q_1}(J_0) \hookrightarrow \ell_\mu^{q_2}(J_0) \quad \text{with } \mu_j := v_j \cdot |\det S_j|^{1-p_2^{-1}} \text{ for } j \in J$$

is bounded, with $\|\eta\| \leq C_{10} \cdot \|\iota\|$ for $C_{10} := \frac{C_2 C_3 C_8 C_9 r_0}{C_1^{(p_2)}}$. But since $\ell_\mu^{q_2}(J_0)$ satisfies the Fatou property, Lemmas 4.8 and 4.6 show that this implies

$$\left\| \left(v_j \cdot |\det S_j|^{1-\frac{1}{p_2}} / u_j \right)_{j \in J_0} \right\|_{\ell_{q_2 \cdot (q_1/q_2)'}'} = \left\| (\mu_j / u_j)_{j \in J_0} \right\|_{\ell_{q_2 \cdot (q_1/q_2)'}'} = \|\eta\| \leq C_{10} \cdot \|\iota\| < \infty.$$

This completes the proof. \square

As before, we now specialize the above lemma to the case where \mathcal{Q} (or more precisely \mathcal{Q}_{I_0}) is relatively \mathcal{P} -moderate. We remark that the following theorem is a slightly generalized version of [23, Theorem 5.3.14] from my PhD thesis.

Theorem 6.23. Let $\emptyset \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ be a *tight* semi-structured L^{p_1} -decomposition covering of \mathcal{O} and let $\mathcal{P} = (P_j)_{j \in J} = (S_j P'_j + c_j)_{j \in J}$ be a *tight* semi-structured L^{p_2} -decomposition covering of \mathcal{O}' , for certain $p_1, p_2 \in (0, \infty]$. Finally, let $q_1, q_2 \in (0, \infty]$ and let $w = (w_i)_{i \in I}$ and $v = (v_j)_{j \in J}$ be \mathcal{Q} -moderate and \mathcal{P} -moderate, respectively.

Choose an arbitrary subset $J_0 \subset J$, set

$$I_0 := \{i \in I \mid J_i \cap J_0 \neq \emptyset\}$$

and assume that the following hold:

- (1) There is $\varepsilon > 0$ and for each $j \in J_0$ some $\xi_j \in \mathbb{R}^d$ with $B_\varepsilon(\xi_j) \subset P'_j$ and $S_j[B_\varepsilon(\xi_j)] + c_j \subset \mathcal{O}$.
- (2) $\mathcal{Q}_{I_0} := (Q_i)_{i \in I_0}$ is almost subordinate to \mathcal{P} .
- (3) \mathcal{Q}_{I_0} is relatively \mathcal{P} -moderate.
- (4) The weight $w|_{I_0}$ is relatively \mathcal{P} -moderate.

Furthermore, set

$$K := \bigcup_{i \in I_0} Q_i \subset \mathcal{O} \cap \mathcal{O}'$$

and assume that the map

$$\iota : \left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})} \right) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2}), f \mapsto f$$

is bounded.

For each $j \in J_0$, let $i_j \in I_j$ be arbitrary¹⁸. Then, we have

$$\left\| \left(\frac{v_j}{w_{i_j}} \cdot |\det T_{i_j}|^{\frac{1}{p_1} - \left(\frac{1}{p_2} - \frac{1}{q_1} \right)_+} - \frac{1}{p_2} \cdot |\det S_j|^{\left(\frac{1}{p_2} - \frac{1}{q_1} \right)_+} \right) \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \leq C \|\iota\| \quad (6.99)$$

for some constant $C > 0$ depending only on

$$d, p_1, p_2, q_1, q_2, k(\mathcal{Q}_{I_0}, \mathcal{P}), C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}), \mathcal{Q}, \mathcal{P}, \varepsilon, \varepsilon_{\mathcal{Q}}, \varepsilon_{\mathcal{P}}, C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}, C_{w|_{I_0}, \mathcal{Q}, \mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}.$$

Here, the L^{p_1}/L^{p_2} -BAPUs $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_j)_{j \in J}$ are those which are used to calculate $\|\iota\|$. ◀

Proof. For brevity, let $k := k(\mathcal{Q}_{I_0}, \mathcal{P})$. Furthermore, set $L := C_{w|_{I_0}, \mathcal{Q}, \mathcal{P}}$, so that we have $w_i \leq L \cdot w_\ell$ for all $i, \ell \in I_0 \cap I_j$ and arbitrary $j \in J$. Moreover, set

$$P_j^{(0)} := S_j[B_\varepsilon(\xi_j)] + c_j \subset P_j \cap \mathcal{O} \quad \forall j \in J_0,$$

with ε, ξ_j as in part (1) of the prerequisites of the current theorem.

As in the proof of Lemma 6.22 (see equation (6.88)), we note

$$\emptyset \neq I_j \subset I_0 \quad \forall j \in J_0.$$

Indeed, we have $\emptyset \neq P_j^{(0)} \subset P_j \cap \mathcal{O}$ and thus $Q_i \cap P_j \neq \emptyset$ for some $i \in I$, i.e. $I_j \neq \emptyset$. Furthermore, every $i \in I_j$ satisfies $j \in J_i \cap J_0 \neq \emptyset$ and thus $i \in I_0$.

Our first goal is to invoke Lemma 2.17 to get $|I_j| = |I_0 \cap I_j| \asymp |\det S_j| / |\det T_{i_j}|$ for all $j \in J_0$. Thus, we first need to verify the assumptions of Lemma 2.17. Note that \mathcal{Q}_{I_0} is relatively \mathcal{P} -moderate and thus in particular relatively \mathcal{P}_{J_0} -moderate. Hence, it remains to verify part (3) of the assumptions of Lemma 2.17, i.e., we need to show

$$\lambda(P_j) \leq C_0 \cdot \lambda \left(\bigcup_{i \in I_0 \cap I_j} Q_i^{r*} \right) \quad (6.100)$$

for all $j \in J_0$ and suitable $r \in \mathbb{N}_0$ and $C_0 > 0$. Note $\lambda(P_j^{(0)}) = C_1 \cdot |\det S_j|$ for all $j \in J_0$ and a suitable constant $C_1 = C_1(d, \varepsilon)$. But Corollary 2.8 yields a constant $C_2 = C_2(d, \mathcal{P})$ satisfying

$$\lambda(P_j) \leq C_2 \cdot |\det S_j| = \frac{C_2}{C_1} \cdot \lambda(P_j^{(0)}).$$

Furthermore, we have $P_j^{(0)} \subset \mathcal{O}$ and hence

$$P_j^{(0)} \subset \bigcup_{\substack{i \in I \\ \text{with } Q_i \cap P_j^{(0)} \neq \emptyset}} Q_i \subset \bigcup_{i \in I_j} Q_i \stackrel{I_j \subseteq I_0}{=} \bigcup_{i \in I_0 \cap I_j} Q_i^{0*}.$$

Thus equation (6.100) is fulfilled for $r := 0$ and $C_0 := \frac{C_2}{C_1}$.

All in all, we have shown that Lemma 2.17 is applicable, so that there are constants

$$C_3 = C_3(d, k(\mathcal{Q}_{I_0}, \mathcal{P}), C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}), \mathcal{Q}, \varepsilon_{\mathcal{Q}}, \mathcal{P}) \quad \text{and} \quad C_4 = C_4(d, C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}), \varepsilon, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{P}})$$

which satisfy

$$|I_0 \cap I_j| \leq C_3 \cdot |\det S_j| \cdot |\det T_{i_j}|^{-1} \quad (6.101)$$

¹⁸We have $S_j[B_\varepsilon(\xi_j)] + c_j \subset \mathcal{O} \cap P_j$, so that there is some $i \in I$ with $Q_i \cap P_j \neq \emptyset$. In particular, $I_j \neq \emptyset$, so that i_j can be chosen as desired.

and

$$|\det T_{i_j}|^{-1} \cdot |\det S_j| \leq C_4 \cdot |I_0 \cap I_j| \quad (6.102)$$

for all $j \in J_0$. Here, we also used that $I_0 \cap I_j = I_j$ is nonempty for all $j \in J_0$, as seen above.

In particular, equation (6.101) shows that $I_0 \cap I_j$ is finite for every $j \in J_0$, so that equation (6.85) from the prerequisites of Lemma 6.22 is satisfied. In fact, using the notation from that lemma, we get

$$\begin{aligned} u_j &= \left\| \left(w_i \cdot |\det T_i|^{1-\frac{1}{p_1}} \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} \\ &\leq LC_5 \cdot w_{i_j} |\det T_{i_j}|^{1-\frac{1}{p_1}} \cdot |I_0 \cap I_j|^{\frac{1}{q_1}} \\ &\leq LC_3^{\frac{1}{q_1}} C_5 \cdot w_{i_j} |\det S_j|^{1/q_1} \cdot |\det T_{i_j}|^{1-\frac{1}{p_1}-\frac{1}{q_1}} < \infty \end{aligned}$$

for all $j \in J_0$ for some constant $C_5 = C_5(p_1, C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}))$. Note that this also holds in case of $1 - \frac{1}{p_1} < 0$, since we have $|\det T_i| \asymp |\det T_{i_j}|$ for all $i \in I_0 \cap I_j$. For brevity, set $C_6 := LC_3^{1/q_1} C_5$.

It is now easy to see that all assumptions of Lemma 6.22 are satisfied, so that we get a constant $C_7 > 0$, which only depends on quantities mentioned in the statement of the theorem, and which satisfies

$$\left\| \left(\frac{v_j}{w_{i_j}} \cdot |\det S_j|^{1-\frac{1}{q_1}-\frac{1}{p_2}} \cdot |\det T_{i_j}|^{\frac{1}{q_1}+\frac{1}{p_1}-1} \right)_{j \in J_0} \right\|_{\ell^q} \leq C_6 \left\| \left(|\det S_j|^{1-\frac{1}{p_2}} \cdot \frac{v_j}{u_j} \right)_{j \in J_0} \right\|_{\ell^q} \leq C_6 C_7 \cdot \|\iota\|, \quad (6.103)$$

where we defined $q := q_2 \cdot (q_1/q_2)'$ for brevity.

To see that this indeed implies the claim, we distinguish two cases:

Case 1: We have $p_2 \in [2, \infty]$ and $q_1 \geq p'_2$. In this case, we get $p_2^\nabla = \min\{p_2, p'_2\} = p'_2$ and $\frac{1}{p_2^\nabla} - \frac{1}{q_1} = \frac{1}{p'_2} - \frac{1}{q_1} \geq 0$. Thus,

$$\left(\frac{1}{p_2^\nabla} - \frac{1}{q_1} \right)_+ = \frac{1}{p'_2} - \frac{1}{q_1} = 1 - \frac{1}{p_2} - \frac{1}{q_1}$$

and hence

$$\begin{aligned} |\det T_{i_j}|^{\frac{1}{p_1} - \left(\frac{1}{p_2^\nabla} - \frac{1}{q_1} \right)_+ - \frac{1}{p_2}} \cdot |\det S_j|^{\left(\frac{1}{p_2^\nabla} - \frac{1}{q_1} \right)_+} &= |\det S_j|^{1-\frac{1}{p_2}-\frac{1}{q_1}} \cdot |\det T_{i_j}|^{\frac{1}{p_1} - \left(1-\frac{1}{p_2}-\frac{1}{q_1} \right) - \frac{1}{p_2}} \\ &= |\det S_j|^{1-\frac{1}{p_2}-\frac{1}{q_1}} \cdot |\det T_{i_j}|^{\frac{1}{p_1}+\frac{1}{q_1}-1}. \end{aligned}$$

Plugging this into the target inequality (6.99), we see that equation (6.103) implies the claim.

Case 2: We have $p_2 \in (0, 2]$ or $q_1 < p'_2$. For brevity, define $s := \left(\frac{1}{p_2^\nabla} - \frac{1}{q_1} \right)_+$ and $t := p_2 \cdot (q_1/p_2)'$. Recall from equation (4.2) that

$$\frac{1}{t} = \left(\frac{1}{p_2} - \frac{1}{q_1} \right)_+.$$

Our first goal is to show $s = \frac{1}{t}$. In case of $p_2 \in (0, 2]$, this is clear, since we have $p_2^\nabla = p_2$. Thus, let us assume $p_2 \in (2, \infty]$. In view of our case distinction, this entails $q_1 < p'_2 = p_2^\nabla$ and hence $s = 0$. But since $p_2 \in (2, \infty]$, we also have $p'_2 < p_2$ and hence $q_1 < p_2$, which entails $\frac{1}{t} = 0 = s$, as desired.

Now, it is easy to see that the assumptions of the current theorem include those of Theorem 6.15. Hence (using the estimate for $\|\Gamma_{\mathcal{Q}}\|_{\ell_w^{q_1} \rightarrow \ell_w^{q_1}}$ and $\|\Gamma_{\mathcal{P}}\|_{\ell_v^{q_2} \rightarrow \ell_v^{q_2}}$ from Lemma 4.13), there is a constant

$$C_8 = C_8(d, p_1, p_2, q_1, q_2, k(\mathcal{Q}_{I_0}, \mathcal{P}), \mathcal{Q}, \varepsilon_{\mathcal{Q}}, \mathcal{P}, C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2})$$

which satisfies $\|\eta\| \leq C_8 \|\iota\|$, with

$$\eta : \ell^{q_1} \left(w_i \cdot |\det T_i|^{p_2^{-1}-p_1^{-1}} \right)_i (I_0) \hookrightarrow \ell_v^{q_2} \left([\ell^{p_2}(I_0 \cap I_j)]_{j \in J} \right).$$

In view of Corollary 5.8 (with $u \equiv 1$ and $r = p_2$, as well as $J_0 = J$), there is hence a suitable constant $C_9 = C_9(q_1, q_2, p_2, k(\mathcal{Q}_{I_0}, \mathcal{P}), \mathcal{P}, C_{v, \mathcal{P}})$ satisfying

$$\left\| \left(v_j \cdot \left\| \left(|\det T_i|^{p_1^{-1}-p_2^{-1}} / w_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{p_2 \cdot (q_1/p_2)'}} \right)_{j \in J} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \leq C_9 \|\eta\| \leq C_8 C_9 \|\iota\|.$$

Hence,

$$\begin{aligned}
 & \left\| \left(\frac{v_j}{w_{i_j}} \cdot |\det T_{i_j}|^{p_1^{-1}-p_2^{-1}-s} \cdot |\det S_j|^s \right)_{j \in J_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\
 & \left(\text{since } s = \frac{1}{t} \right) = \left\| \left(\frac{v_j}{w_{i_j}} \cdot |\det T_{i_j}|^{p_1^{-1}-p_2^{-1}} \cdot |\det S_j|^{\frac{1}{t}} / |\det T_{i_j}|^{\frac{1}{t}} \right)_{j \in J_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\
 & \text{(eq. (6.102))} \leq C_4^{1/t} \cdot \left\| \left(\frac{v_j}{w_{i_j}} \cdot |\det T_{i_j}|^{p_1^{-1}-p_2^{-1}} \cdot \|(1)_{i \in I_0 \cap I_j}\|_{\ell^t} \right)_{j \in J_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\
 & \leq LC_4^{1/t} C_{10} \cdot \left\| \left(v_j \cdot \left\| (|\det T_{i_j}|^{p_1^{-1}-p_2^{-1}} / w_{i_j})_{i \in I_0 \cap I_j} \right\|_{\ell^{p_2 \cdot (q_1/p_2)'}} \right)_{j \in J} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\
 & \leq LC_4^{1/t} C_8 C_9 C_{10} \cdot \|\iota\|
 \end{aligned}$$

for some constant $C_{10} = C_{10} (C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}), p_1, p_2)$. This is precisely the desired estimate. \square

7. SUMMARY OF EMBEDDING RESULTS

While we strove for maximal generality in the preceding sections, our aim in this section is ease of applicability, even if this makes our results slightly less general. Thus, those readers who are interested in embeddings between decomposition spaces which are highly “incompatible”—e.g. there is no subordinateness or no relative moderateness or we have neither $\mathcal{O} \subset \mathcal{O}'$, nor $\mathcal{O}' \subset \mathcal{O}$ —or for which the “global components” are not just weighted ℓ^q -spaces, are encouraged to browse the results from the preceding sections and not only this one. But for the most common cases, the results in this section are easily applicable and yield optimal results.

To simplify notation, we will employ the following general assumptions in this section:

Assumption 7.1. *Let $\emptyset \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open and let $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ and $\mathcal{P} = (S_j P'_j + c_j)_{j \in J}$ be two open, tight, semi-structured coverings of \mathcal{O} and \mathcal{O}' , respectively. Let $w = (w_i)_{i \in I}$ and $v = (v_j)_{j \in J}$ be \mathcal{Q} -moderate and \mathcal{P} -moderate, respectively and let $p_1, p_2, q_1, q_2 \in (0, \infty]$.*

Furthermore, assume that $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_j)_{j \in J}$ are L^p -BAPUs for \mathcal{Q} or \mathcal{P} , respectively, simultaneously for all $p \in (0, \infty]$.

Finally, assume that the (quasi)-norms $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})}$ and $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})}$ are calculated using the BAPUs Φ and Ψ , respectively.

Remark. Note that the preceding assumptions—excluding the moderateness of w, v —are always fulfilled if \mathcal{Q} and \mathcal{P} are almost structured coverings (cf. Definition 2.5 and the ensuing remark), for a proper choice of Φ, Ψ (cf. Theorem 3.19). \blacklozenge

We will always be interested in sufficient or necessary conditions for an embedding of the form

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2}),$$

under varying (additional) assumptions on \mathcal{Q}, \mathcal{P} .

We begin with the case in which \mathcal{P} is almost subordinate to \mathcal{Q} . We remark that the following result is a generalized version of [23, Theorem 5.4.1] from my PhD thesis.

Theorem 7.2. In addition to our standing assumptions, assume that \mathcal{P} is almost subordinate to \mathcal{Q} . Note that this entails $\mathcal{O}' \subset \mathcal{O}$.

For $i \in I$, let

$$J_i := \{j \in J \mid P_j \cap Q_i \neq \emptyset\}$$

and for $r \in (0, \infty]$ and $\ell \in \{1, 2\}$, define

$$K_{r,\ell} := \left\| \left(w_i^{-1} \cdot \left\| \left(v_j / u_{i,j}^{(\ell)} \right)_{j \in J_i} \right\|_{\ell^{q_2 \cdot (r/q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \in [0, \infty],$$

with $u_{i,j}^{(1)} := |\det S_j|^{\frac{1}{p_2} - \frac{1}{p_1}}$ for $j \in J$,

$$u_{i,j}^{(2)} := \begin{cases} |\det S_j|^{\frac{1}{p_2} - 1} \cdot |\det T_i|^{1 - \frac{1}{p_1}}, & \text{if } p_1 < 1, \\ |\det S_j|^{\frac{1}{p_2} - \frac{1}{p_1}}, & \text{if } p_1 \geq 1. \end{cases}$$

Then, the following hold:

- (1) If we have $p_1 \leq p_2$ and if $K_{p_1^\Delta, 2} < \infty$, then

$$\iota : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2}), f \mapsto f|_{C_c^\infty(\mathcal{O}')}$$

is well-defined and bounded, with $\|\iota\| \leq C_1 \cdot K_{p_1^\Delta, 2}$ for some constant

$$C_1 = C_1(d, p_1, p_2, q_1, q_2, \mathcal{Q}, \mathcal{P}, k(\mathcal{P}, \mathcal{Q}), C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_1}, C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}).$$

- (2) Conversely, if the map

$$\theta : \left(C_c^\infty(\mathcal{O}'), \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})} \right) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2}), f \mapsto f$$

is bounded, then we have $p_1 \leq p_2$ and $K_{p_1, 1} \leq C_2 \cdot \|\theta\| < \infty$ for some constant

$$C_2 = C_2(d, p_1, p_2, q_1, q_2, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{P}}, k(\mathcal{P}, \mathcal{Q}), C_{\mathcal{Q}, \Phi, p_1}, C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}).$$

- (3) Under the assumptions of the preceding point, if $p_1 = p_2$, then $K_{2, 1} \leq C_3 \cdot \|\theta\| < \infty$ for some constant

$$C_3 = C_3(d, p_1, q_1, q_2, \mathcal{Q}, \mathcal{P}, k(\mathcal{P}, \mathcal{Q}), C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}, C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}).$$

- (4) Finally, if we have $\mathcal{O} = \mathcal{O}'$ and if \mathcal{P} and v are relatively \mathcal{Q} -moderate, then we have the following equivalence (with ι, θ as above): Setting $s := \left(\frac{1}{q_2} - \frac{1}{p_1^\Delta} \right)_+$ and choosing for each $i \in I$ some $j_i \in J$ with $Q_i \cap P_{j_i} \neq \emptyset$, we have:

ι well-defined and bounded

$\iff \theta$ well-defined and bounded

$$\iff p_1 \leq p_2 \text{ and } K := \left\| \left(\frac{v_{j_i}}{w_i} \cdot |\det T_i|^s \cdot |\det S_{j_i}|^{\frac{1}{p_1} - \frac{1}{p_2} - s} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} < \infty.$$

Furthermore, given $p_1 \leq p_2$, we have

$$\|\iota\| \asymp \|\theta\| \asymp K,$$

where the implied constants only depend on

$$d, p_1, p_2, q_1, q_2, \mathcal{Q}, \varepsilon_{\mathcal{Q}}, \mathcal{P}, \varepsilon_{\mathcal{P}}, k(\mathcal{P}, \mathcal{Q}), C_{\text{mod}}(\mathcal{P}, \mathcal{Q}), C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}, C_{v, \mathcal{P}, \mathcal{Q}}, C_{\mathcal{Q}, \Phi, p_1}. \quad \blacktriangleleft$$

Proof. In the whole proof, let us set $Y := \ell_w^{q_1}(I)$ and $Z := \ell_v^{q_2}(J)$, as well as $J_0 := J$. We now prove each statement individually.

1. Remark 5.13 shows that the embedding

$$\eta : Y \left(\left[\ell_{u^{(2)}}^{p_1^\Delta}(J_0 \cap J_i) \right]_{i \in I} \right) \hookrightarrow Z|_{J_0}$$

satisfies

$$\|\eta\| \asymp K_{p_1^\Delta, 2},$$

where the implied constant only depends on $p_1, q_1, q_2, C_{w, \mathcal{Q}}, \mathcal{Q}, k(\mathcal{P}, \mathcal{Q})$.

Hence, boundedness of ι , with

$$\|\iota\| \leq C \cdot \|\eta\| \leq C' \cdot K_{p_1^\Delta, 2} < \infty$$

for suitable constants C, C' as in the statement of the theorem, follows directly from Corollary 5.11. Here, we also used Lemma 4.13 to estimate $\|\Gamma_{\mathcal{Q}}\|_{\ell_w^{q_1} \rightarrow \ell_w^{q_1}}$ and $\|\Gamma_{\mathcal{P}}\|_{\ell_v^{q_2} \rightarrow \ell_v^{q_2}}$ by quantities involving only $q_1, q_2, \mathcal{Q}, \mathcal{P}, C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}$.

2. First, note that $C_c^\infty(\mathcal{O}') \subset C_c^\infty(\mathcal{O})$ and hence $C_c^\infty(\mathcal{O}) \subset \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \cap \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$, so that θ is always well-defined, but not necessarily bounded. Of course, in the present case, θ is bounded by assumption.

Now, it is easy to see that Lemma 6.1 is applicable with $K := \mathcal{O}' \subset \mathcal{O} \cap \mathcal{O}'$. Picking any $j \in J$ and any $\xi \in P_j \subset \mathcal{O}' \subset \mathcal{O}$, there is some $i \in I$ with $\xi \in Q_i$. Hence, $\xi \in K^\circ \cap Q_i^\circ \cap P_j^\circ$, since by our standing assumptions, \mathcal{Q} and \mathcal{P} are open coverings. Finally, $\delta_i \in Y = \ell_w^{q_1}(I)$ for arbitrary $i \in I$, so that Lemma 6.1 yields $p_1 \leq p_2$, as claimed.

Next, note that our assumptions (in particular, boundedness of θ) imply that Theorem 6.13 (with Y, Z, J_0 as above) is applicable. Since $Z = \ell_v^{q_2}(J)$ satisfies the Fatou property, this yields boundedness of

$$\eta : \ell_w^{q_1}([\ell^{p_1}(J_i)]_{i \in I}) = Y([\ell^{p_1}(J_i \cap J_0)]_{i \in I}) \hookrightarrow Z_{|\det S_j|^{p_1^{-1}-p_2^{-1}}} = \ell_w^{q_2} \left[v_j \cdot |\det S_j|^{p_1^{-1}-p_2^{-1}} \right]_j (J),$$

with $\|\eta\| \leq C \cdot \|\theta\|$ for some constant

$$C = C(d, p_1, p_2, q_2, k(\mathcal{P}, \mathcal{Q}), \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, C_{v, \mathcal{P}}).$$

Here, we used that the triangle constant C_Z for $Z = \ell_v^{q_2}(J)$ only depends on q_2 and that Lemma 4.13 allows to estimate $\|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}$ in terms of $q_2, \mathcal{P}, C_{v, \mathcal{P}}$.

But in view of Corollary 5.12 (with $r = p_1$, $u^{(1)} \equiv 1$ and $u^{(2)} \equiv 1$, as well as $(v_j \cdot |\det S_j|^{p_1^{-1}-p_2^{-1}})_{j \in J}$ instead of v), this entails

$$\left\| \left(w_i^{-1} \cdot \left\| \left(v_j \cdot |\det S_j|^{p_1^{-1}-p_2^{-1}} \right)_{j \in J_i} \right\|_{\ell_{q_2 \cdot (p_1/q_2)'}} \right)_{i \in I} \right\|_{\ell_{q_2 \cdot (q_1/q_2)'}} \leq C' \cdot \|\eta\| \leq CC' \cdot \|\theta\|$$

for some constant $C' = C'(p_1, q_1, q_2, C_{w, \mathcal{Q}}, N_{\mathcal{Q}}, k(\mathcal{P}, \mathcal{Q}))$. But the left-hand side of this estimate is simply $K_{p_1, 1}$.

3. In this case, Theorem 6.17 (and the ensuing remark) show that

$$\eta : \ell_w^{q_1}([\ell^2(J_i)]_{i \in I}) = Y([\ell^2(J_0 \cap J_i)]_{i \in I}) \hookrightarrow Z = \ell_v^{q_2}(J)$$

is well-defined and bounded, with $\|\eta\| \leq C \cdot \|\theta\|$ for some constant

$$C = C(d, p_1, q_1, q_2, \mathcal{Q}, \mathcal{P}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}, C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}).$$

Here, we again used that the triangle constant C_Z of $Z = \ell_v^{q_2}(J)$ only depends on q_2 and that Lemma 4.13 allows to estimate $\|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}$ and $\|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}$ in terms of $q_1, q_2, \mathcal{Q}, \mathcal{P}, C_{w, \mathcal{Q}}$ and $C_{v, \mathcal{P}}$.

The remainder of the proof proceeds precisely as in the previous case (with $r = 2$ instead of $r = p_1$), noting that we have $(v_j \cdot |\det S_j|^{p_1^{-1}-p_2^{-1}})_{j \in J} = v$, since $p_1 = p_2$.

4. Let us first prove the estimate $K \lesssim \|\theta\|$. To do this, it suffices to verify that the assumptions of Theorem 6.21 are met; these consist of the following:

- (1) $\mathcal{P}_{J_0} = \mathcal{P}$ is relatively \mathcal{Q} -moderate. This holds by assumption.
- (2) The weight $v = v|_{J_0}$ is relatively \mathcal{Q} -moderate. Again, this holds by assumptions.
- (3) There is $r \in \mathbb{N}_0$ and some $C_0 > 0$ satisfying

$$\lambda(Q_i) \leq C_0 \cdot \lambda \left(\bigcup_{j \in J_0 \cap J_i} P_j^{r*} \right).$$

But since we are assuming $\mathcal{O} = \mathcal{O}'$, we have $Q_i \subset \mathcal{O} = \mathcal{O}' = \bigcup_{j \in J} P_j$. Since we also have $J_0 = J$, we see that the preceding estimate is satisfied for $C_0 = 1$ and $r = 0$.

- (4) The assumptions of Lemma 6.19 are satisfied. This is indeed the case, as we will verify now.

First, $\mathcal{P} = \mathcal{P}_{J_0}$ is almost subordinate to \mathcal{Q} , as required in Lemma 6.19. Finally, we want to verify that θ satisfies the properties of the map ι from that lemma. Hence, we have to verify

$$\langle \theta f, \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in C_c^\infty(\mathcal{O} \cap \mathcal{O}') \text{ and } f \in \mathcal{D}_K^{\mathcal{Q}, p_1, \ell_w^{q_1}(I)},$$

for a certain set $K \subset \mathcal{O}$, which is defined in the statement of Lemma 6.19, but whose precise definition is unimportant for us. Indeed, we have $K \subset \mathcal{O} = \mathcal{O}'$ and hence $\theta f = f$ for all $f \in \mathcal{D}_K^{\mathcal{Q}, p_1, \ell_w^{q_1}(I)} \subset C_c^\infty(\mathcal{O}')$, which easily implies the desired identity from above.

All in all, we see that Lemma 6.19, and thus also Theorem 6.21 are applicable, so that we get $K \lesssim \|\theta\|$, with a constant as in the statement of the current theorem. In view of the second part of the current theorem, it is clear that boundedness of θ also entails $p_1 \leq p_2$.

Next, if ι is bounded, it is clear that $\iota|_{C_c^\infty(\mathcal{O}')}$ satisfies the properties required of θ , so that we get $K \lesssim \|\iota|_{C_c^\infty(\mathcal{O}')} \| \leq \|\iota\|$ and $p_1 \leq p_2$.

Hence, it remains to show that if $p_1 \leq p_2$ and if K is finite, then ι is well-defined and bounded with $\|\iota\| \lesssim K$. In view of the first part of the theorem, it is thus sufficient to verify $K_{p_1^\Delta, 2} \lesssim K$. But this is a direct consequence of Remark 5.13, see in particular equation (5.24). \square

One easy, but nevertheless frequently useful special case is when \mathcal{Q} and \mathcal{P} coincide.

Corollary 7.3. *In addition to our standing assumptions, assume that $\mathcal{Q} = \mathcal{P}$. Then, the identity map*

$$\iota : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_2}, \ell_v^{q_2}), f \mapsto f$$

is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if

$$K := \left\| \left(|\det T_i|^{p_1^{-1} - p_2^{-1}} \cdot \frac{v_i}{w_i} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'} } < \infty.$$

More precisely, in case of $p_1 \leq p_2$, we have $\|\iota\| \asymp K$, where the implied constants only depend on

$$d, p_1, p_2, q_1, q_2, \mathcal{Q}, \varepsilon_{\mathcal{Q}}, C_{w, \mathcal{Q}}, C_{v, \mathcal{Q}}, C_{\mathcal{Q}, \Phi, p_1}.$$

◀

Proof. We apply part (4) of Theorem 7.2 (with $\mathcal{Q} = \mathcal{P}$). First of all, $\mathcal{P}(= \mathcal{Q})$ is almost subordinate to \mathcal{Q} with $k(\mathcal{P}, \mathcal{Q}) = 0$, since we have $P_j = Q_j \subset Q_j^{0*}$ for all $j \in J = I$. Next, note that $\mathcal{P}(= \mathcal{Q})$ and v are clearly relatively \mathcal{Q} -moderate, with $C_{\text{mod}}(\mathcal{P}, \mathcal{Q}) = C(d, C_{\mathcal{Q}})$ and $C_{v, \mathcal{P}, \mathcal{Q}} = C(C_{v, \mathcal{Q}})$, cf. equation (3.11). Furthermore, \mathcal{Q} and $\mathcal{P}(= \mathcal{Q})$ both cover the same set \mathcal{O} , so that part (4) of Theorem 7.2 is indeed applicable.

Thus, we see that ι is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if

$$\tilde{K} := \left\| \left(\frac{v_{j_i}}{w_i} \cdot |\det T_i|^s \cdot |\det S_{j_i}|^{\frac{1}{p_1} - \frac{1}{p_2} - s} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)' } }$$

is finite, where $s = \left(\frac{1}{q_2} - \frac{1}{p_1^\Delta} \right)_+$ and where for each $i \in I$, some $j_i \in J_i$ can be arbitrarily selected. Because of $\mathcal{Q} = \mathcal{P}$, we can simply choose $j_i := i$, so that we get

$$\tilde{K} = \left\| \left(|\det T_i|^{p_1^{-1} - p_2^{-1}} \cdot \frac{v_i}{w_i} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)' } }.$$

Now, all claims follow from part (4) of Theorem 7.2. \square

As our next result, compared to Theorem 7.2, we consider the “reverse” case, in which \mathcal{Q} is almost subordinate to \mathcal{P} . We remark that the following theorem is an improved version of [23, Theorem 5.4.4] from my PhD thesis.

Theorem 7.4. *In addition to our standing assumptions, assume that \mathcal{Q} is almost subordinate to \mathcal{P} . Note that this entails $\mathcal{O} \subset \mathcal{O}'$.*

For $j \in J$, let

$$I_j := \{i \in I \mid Q_i \cap P_j \neq \emptyset\}$$

and for $r \in (0, \infty]$, define

$$K_r := \left\| \left(v_j \cdot \left\| \left(|\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}} / w_i \right)_{i \in I_j} \right\|_{\ell^{r \cdot (q_1/r)' } } \right)_{j \in J} \right\|_{\ell^{q_2 \cdot (q_1/q_2)' } } \in [0, \infty]. \quad (7.1)$$

Then, the following hold:

- (1) If we have $p_1 \leq p_2$ and if $K_{p_2^\vee} < \infty$, then

$$\iota : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2}), f \mapsto \sum_{i \in I} \varphi_i f$$

is well-defined and bounded, with $\|\iota\| \leq C_1 \cdot K_{p_2^\vee}$ for some constant

$$C_1 = C_1(d, p_1, p_2, q_1, q_2, \mathcal{Q}, \mathcal{P}, k(\mathcal{Q}, \mathcal{P}), C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}, C_{v, \mathcal{P}}).$$

Furthermore,

- (a) for each $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \leq \mathcal{D}'(\mathcal{O})$, the distribution $\iota f \in \mathcal{D}'(\mathcal{O}')$ is an extension of f onto $C_c^\infty(\mathcal{O}')$,
 (b) for each $f \in C_c^\infty(\mathcal{O})$, we have $\iota f = f$.
 (2) Conversely, if the map

$$\theta : (C_c^\infty(\mathcal{O}), \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})}) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2}), g \mapsto g$$

is bounded, then we have $p_1 \leq p_2$ and

$$K_s \leq C_2 \cdot \|\theta\| < \infty \quad \text{with} \quad s := \begin{cases} p_2, & \text{if } p_2 < \infty, \\ 1 = p_2^\vee, & \text{if } p_2 = \infty \end{cases}$$

and some constant

$$C_2 = C_2(d, p_1, p_2, q_1, q_2, \mathcal{Q}, \varepsilon_{\mathcal{Q}}, \mathcal{P}, k(\mathcal{Q}, \mathcal{P}), C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}, C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}).$$

- (3) Under the assumptions of the preceding point, if $p_1 = p_2 =: p$ and

$$s := \begin{cases} 1 & \text{if } p = \infty, \\ \min\{2, p\}, & \text{if } p < \infty, \end{cases}$$

then $K_s \leq C_3 \cdot \|\theta\| < \infty$ for some constant

$$C_3 = C_3(d, p_1, q_1, q_2, \mathcal{Q}, \mathcal{P}, k(\mathcal{Q}, \mathcal{P}), C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}, C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}).$$

- (4) Finally, if we have $\mathcal{O} = \mathcal{O}'$ and if \mathcal{Q} and w are relatively \mathcal{P} -moderate, then we have the following equivalence (with ι, θ as above): Setting $s := \left(\frac{1}{p_2} - \frac{1}{q_1}\right)_+$ and choosing for each $j \in J$ some $i_j \in I$ with $P_j \cap Q_{i_j} \neq \emptyset$, we have

$$\begin{aligned} & \iota \text{ well-defined and bounded} \\ & \iff \theta \text{ well-defined and bounded} \\ & \iff p_1 \leq p_2 \text{ and } K := \left\| \left(\frac{v_j}{w_{i_j}} \cdot |\det T_{i_j}|^{\frac{1}{p_1} - \frac{1}{p_2} - s} \cdot |\det S_j|^s \right)_{j \in J} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'} } < \infty. \end{aligned}$$

Furthermore, given $p_1 \leq p_2$, we have

$$\|\iota\| \asymp \|\theta\| \asymp K,$$

where the implied constants only depend on

$$d, p_1, p_2, q_1, q_2, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{Q}}, \varepsilon_{\mathcal{P}}, k(\mathcal{Q}, \mathcal{P}), C_{\text{mod}}(\mathcal{Q}, \mathcal{P}), C_{w, \mathcal{Q}}, C_{w, \mathcal{P}}, C_{v, \mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}. \quad \blacktriangleleft$$

Proof. For the whole proof, set $Y := \ell_w^{q_1}(I)$, $Z := \ell_v^{q_2}(J)$ and $I_0 := I$. We now prove each statement individually.

1. Remark 5.9 shows that the embedding η from Corollary 5.7 (with $I_0 = I$) satisfies

$$\|\eta\| \asymp \left\| \left(v_j \cdot \left\| |\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}} / w_i \right\|_{\ell^{p_2^\vee \cdot (q_1/p_2)'}} \right)_{i \in I_j} \right\|_{j \in J} \Big\|_{\ell^{q_2 \cdot (q_1/q_2)'}} = K_{p_2^\vee},$$

where the implied constant only depends on $q_1, q_2, p_2, N_{\mathcal{P}}, k(\mathcal{Q}, \mathcal{P}), C_{v, \mathcal{P}}$.

Since we also assume $p_1 \leq p_2$, Corollary 5.7 shows that the stated map ι is well-defined and bounded and satisfies property (1a), since $I_0 = I$. Furthermore, since $\Phi = (\varphi_i)_{i \in I}$ is a locally finite partition of unity on \mathcal{O} , we have for $g \in C_c^\infty(\mathcal{O})$ that $g = \sum_{i \in I} \varphi_i g = \iota g$, where only finitely many terms do not vanish.

Finally, Corollary 5.7 even yields $\|\iota\| \lesssim \|\eta\|$, where the implied constant only depends on quantities mentioned in the current theorem—at least, once we use Lemma 4.13 to estimate $\|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}$ in terms of $q_2, \mathcal{P}, C_{v, \mathcal{P}}$.

2. The present assumptions show that Lemma 6.1 (with $K = \mathcal{O} \subset \mathcal{O} \cap \mathcal{O}'$) is applicable. Now, for $i \in I$ and arbitrary $\xi \in Q_i \subset \mathcal{O} \subset \mathcal{O}'$, we get $\xi \in Q_i \cap P_j \neq \emptyset$ for some $j \in J$. But since \mathcal{Q}, \mathcal{P} are open coverings, we get $K^\circ \cap Q_i^\circ \cap P_j^\circ \neq \emptyset$, so that Lemma 6.1 yields $p_1 \leq p_2$.

Next, note that $Z = \ell_v^{q_2}(J)$ satisfies the Fatou property and that our assumptions imply that Theorem 6.15 is applicable, so that the embedding

$$\eta : (Y|_{I_0})_{|\det T_i|^{p_2^{-1}-p_1^{-1}}} \hookrightarrow Z \left([\ell^s(I_0 \cap I_j)]_{j \in J} \right),$$

i.e.

$$\eta : \ell^{q_1} \left(|\det T_i|^{p_2^{-1}-p_1^{-1}} \cdot w_i \right)_i (I) \hookrightarrow \ell_v^{q_2} \left([\ell^s(I_j)]_{j \in J} \right)$$

is well-defined and bounded, with $\|\eta\| \leq C \cdot \|\theta\|$ for some constant

$$C = C(d, p_1, p_2, q_1, q_2, \mathcal{Q}, \varepsilon_{\mathcal{Q}}, \mathcal{P}, k(\mathcal{Q}, \mathcal{P}), C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}, C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}).$$

Here, we again used Lemma 4.13 to estimate $\|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}$ and $\|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}$ in terms of $q_1, q_2, \mathcal{Q}, \mathcal{P}, C_{w, \mathcal{Q}}$ and $C_{v, \mathcal{P}}$ and we also used that the triangle constant C_Z of $Z = \ell_v^{q_2}(J)$ only depends on q_2 .

But in view of Corollary 5.8 (with $J_0 = J$, $u \equiv 1$, $r = s$ and $(|\det T_i|^{p_2^{-1}-p_1^{-1}} \cdot w_i)_{i \in I}$ instead of w), boundedness of η entails

$$\left\| \left(v_j \cdot \left\| \left(|\det T_i|^{p_1^{-1}-p_2^{-1}} / w_i \right)_{i \in I_j} \right\|_{\ell^{s \cdot (q_1/s)'}} \right)_{j \in J} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \leq C' \|\eta\| \leq C \|\theta\|$$

for some constant $C' = C'(q_1, q_2, p_2, \mathcal{P}, k(\mathcal{Q}, \mathcal{P}), C_{v, \mathcal{P}})$. But the left-hand side of this estimate is simply K_s .

3. In this case, Theorem 6.18 (and the ensuing remark) show that

$$\eta : Y|_{I_0} \hookrightarrow Z \left([\ell^s(I_0 \cap I_j)]_{j \in J} \right)$$

is well-defined and bounded, with $\|\eta\| \leq C \cdot \|\theta\|$ for some constant

$$C = C(d, p_1, q_1, q_2, \mathcal{Q}, \mathcal{P}, k(\mathcal{Q}, \mathcal{P}), C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}, C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}).$$

Here, we again used that the triangle constant C_Z of $Z = \ell_v^{q_2}(J)$ only depends on q_2 and that Lemma 4.13 allows to estimate $\|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}$ and $\|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}$ in terms of $q_1, q_2, \mathcal{Q}, \mathcal{P}, C_{w, \mathcal{Q}}$ and $C_{v, \mathcal{P}}$.

The rest of the proof is as in the previous case, noting that we have $(|\det T_i|^{p_2^{-1}-p_1^{-1}} \cdot w_i)_{i \in I} = w$, since $p_1 = p_2$.

4. Under the present assumptions, let us first assume that θ is well-defined and bounded. As seen above, this yields $p_1 \leq p_2$, so that it suffices to show $K \lesssim \|\theta\|$, with an implied constant as stated. To see this, we want to apply Theorem 6.23, with $J_0 := J$. In the notation of that theorem, this implies

$$\begin{aligned} I_0 &= \{i \in I \mid J_i \cap J_0 \neq \emptyset\} \\ &= \{i \in I \mid J_i \neq \emptyset\} \\ &= \{i \in I \mid \exists j \in J : Q_i \cap P_j \neq \emptyset\} \end{aligned}$$

$$(\text{since } \mathcal{P} \text{ covers } \mathcal{O}') = \{i \in I \mid Q_i \cap \mathcal{O}' \neq \emptyset\} = I,$$

where the last step used that we have $\mathcal{O}' = \mathcal{O}$ and that each Q_i is nonempty. Now, let us verify assumptions (1)–(4) of Theorem 6.23:

- (1) Since $\mathcal{P} = (S_j P'_j + c_j)_{j \in J}$ is a tight semi-structured covering, there is for $\varepsilon := \varepsilon_{\mathcal{P}}$ for each $j \in J = J_0$ some $\xi_j = c_j \in \mathbb{R}^d$ with $B_\varepsilon(\xi_j) \subset P'_j$. Particularly, $S_j[B_\varepsilon(\xi_j)] + c_j \subset P_j \subset \mathcal{O}' = \mathcal{O}$, as desired.
- (2) By assumption, $\mathcal{Q}_{I_0} = \mathcal{Q}$ is almost subordinate to \mathcal{P} .
- (3) By assumption, $\mathcal{Q}_{I_0} = \mathcal{Q}$ is relatively \mathcal{P} -moderate.
- (4) By assumption, the weight $w = w|_{I_0}$ is relatively \mathcal{P} -moderate.

Finally, in view of $\bigcup_{i \in I_0} Q_i = \mathcal{O}$ we see that the map ι from Theorem 6.23 coincides with θ from the current theorem. In particular, ι (as in Theorem 6.23) is bounded, so that Theorem 6.23 yields $K \leq C \|\theta\| < \infty$ for some constant

$$C = C(d, p_1, p_2, q_1, q_2, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{Q}}, \varepsilon_{\mathcal{P}}, k(\mathcal{Q}, \mathcal{P}), C_{\text{mod}}(\mathcal{Q}, \mathcal{P}), C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}, C_{w, \mathcal{Q}, \mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}),$$

as desired.

Now, by the properties of ι shown in the first part of the present theorem, we see that θ is a restriction of ι . In particular, if ι is bounded, then so is θ , so that we get $p_1 \leq p_2$ and $K \lesssim \|\theta\| \leq \|\iota\|$.

Finally, we need to show that $p_1 \leq p_2$ and $K < \infty$ imply boundedness of ι . In view of the first part of the current theorem, it suffices to show $K_{p_2^\vee} \lesssim K$. But this is an immediate consequence of Remark 5.9 (with $J_0 = J$, $u_i = |\det T_i|^{p_1^{-1} - p_2^{-1}}$ and $I_0 = I$), once we verify its assumptions, which are the assumptions of the second part of Corollary 5.8. But the only assumptions which are not obviously implied by the present ones are relative moderateness of $u = \left(|\det T_i|^{p_1^{-1} - p_2^{-1}}\right)_{i \in I}$ —which follows from relative \mathcal{P} -moderateness of \mathcal{Q} —and the existence of $s \in \mathbb{N}_0$ and $C_0 > 0$ satisfying

$$\lambda(P_j) \leq C_0 \cdot \lambda\left(\bigcup_{i \in I_0 \cap I_j} Q_i^{s*}\right)$$

for all $j \in J$. But since we have $P_j \subset \mathcal{O}' = \mathcal{O}$ and since \mathcal{Q} covers \mathcal{O} , we have $P_j \subset \bigcup_{i \in I_j} Q_i$. Since we have $I_0 = I$, we can thus take $C_0 = 1$ and $s = 0$.

Finally, the implicit constant in the estimate $K_{p_2^\vee} \lesssim K$ provided by Remark 5.9 only depends on

$$d, q_1, q_2, p_1, p_2, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{Q}}, \varepsilon_{\mathcal{P}}, k(\mathcal{Q}, \mathcal{P}), C_{\text{mod}}(\mathcal{Q}, \mathcal{P}), C_{w, \mathcal{Q}, \mathcal{P}}, C_{v, \mathcal{P}}. \quad \square$$

Our final theorem in this summary is concerned with the “intermediate” case in which some part of \mathcal{Q} is almost subordinate to \mathcal{P} and vice versa.

Strictly speaking, the two preceding theorems are special cases of this theorem. We nevertheless stated them separately, since their application is *much* more convenient than invoking the following theorem, which is an improved version of [23, Theorem 5.4.5] from my PhD thesis.

Theorem 7.5. In addition to our standing assumptions, suppose that we have

$$\emptyset \neq \mathcal{O} \cap \mathcal{O}' = A \cup B$$

for certain sets A, B and such that the following additional properties are satisfied:

- (1) With $I_A := \{i \in I \mid Q_i \cap A \neq \emptyset\}$, the family $\mathcal{Q}_{I_A} = (Q_i)_{i \in I_A}$ is almost subordinate to \mathcal{P} .
- (2) With $J_B := \{j \in J \mid P_j \cap B \neq \emptyset\}$, the family $\mathcal{P}_{J_B} = (P_j)_{j \in J_B}$ is almost subordinate to \mathcal{Q} .

For $r \in (0, \infty]$ and $\ell \in \{1, 2\}$, define

$$K_r^{(1)} := \left\| \left(v_j \cdot \left\| \left(|\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}} / w_i \right)_{i \in I_j \cap I_A} \right\|_{\ell^{r \cdot (q_1/r)'} } \right)_{j \in J} \right\|_{\ell^{q_2 \cdot (q_1/q_2)' } } \in [0, \infty]$$

$$K_r^{(2, \ell)} := \left\| \left(w_i^{-1} \cdot \left\| \left(v_j / u_{i,j}^{(\ell)} \right)_{j \in J_i \cap J_B} \right\|_{\ell^{q_2 \cdot (r/q_2)' } } \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)' } } \in [0, \infty],$$

with

$$u_{i,j}^{(1)} := |\det S_j|^{p_2^{-1} - p_1^{-1}} \quad \text{and} \quad u_{i,j}^{(2)} := \begin{cases} |\det S_j|^{p_2^{-1} - 1} \cdot |\det T_i|^{1 - p_1^{-1}}, & \text{if } p_1 < 1, \\ |\det S_j|^{p_2^{-1} - p_1^{-1}}, & \text{if } p_1 \geq 1. \end{cases}$$

Then the following hold:

- (1) If $p_1 \leq p_2$ and if $K_{p_2^\vee}^{(1)} < \infty$ and $K_{p_1^\Delta}^{(2,2)} < \infty$, then there is a bounded linear map

$$\iota : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$$

with $\|\iota\| \lesssim K_{p_2^\vee}^{(1)} + K_{p_1^\Delta}^{(2,2)}$, where the implied constant only depends on

$$d, p_1, p_2, q_1, q_2, \mathcal{Q}, \mathcal{P}, k(\mathcal{Q}_{I_A}, \mathcal{P}), k(\mathcal{P}_{J_B}, \mathcal{Q}), C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_1}$$

and with the following additional properties:

- (a) For $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$ and $g \in C_c^\infty(\mathcal{O} \cap \mathcal{O}')$, we have

$$\langle f, g \rangle = \langle \iota f, g \rangle.$$

In particular, if $\mathcal{O} = \mathcal{O}'$, then $\iota f = f$ for all $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \leq \mathcal{D}'(\mathcal{O})$.

- (b) If $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$ is given by (integration against) a measurable function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ with $f \in L_{\text{loc}}^1(\mathcal{O} \cup \mathcal{O}')$ and with $f = 0$ almost everywhere on $\mathcal{O}' \setminus \mathcal{O}$, then $\iota f = f$.
 (c) In particular, $\iota f = f$ for all $f \in C_c^\infty(\mathcal{O} \cap \mathcal{O}')$.
 (d) For $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$, we have $\text{supp } \iota f \subset \mathcal{O}' \cap \overline{\text{supp } f}$.
- (2) Conversely, if

$$\theta : \left(C_c^\infty(\mathcal{O} \cap \mathcal{O}'), \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})} \right) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2}), f \mapsto f$$

is bounded, then we have

- (a) $p_1 \leq p_2$,
 (b) $K_s^{(1)} \lesssim \|\theta\|$ with $s = \begin{cases} 1, & \text{if } p_2 = \infty, \\ p_2, & \text{otherwise,} \end{cases}$
 (c) $K_{p_1}^{(2,1)} \lesssim \|\theta\|$.

Here, all implied constants only depend on

$$d, p_1, p_2, q_1, q_2, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{Q}}, \varepsilon_{\mathcal{P}}, k(\mathcal{Q}_{I_A}, \mathcal{P}), k(\mathcal{P}_{J_B}, \mathcal{Q}), C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}.$$

- (3) Under the assumptions of the previous case, if additionally $p_1 = p_2$, we also have $K_2^{(1)} \lesssim \|\theta\|$ and $K_2^{(2,1)} \lesssim \|\theta\|$, where the implied constants only depend on

$$d, p_1, q_1, q_2, \mathcal{Q}, \mathcal{P}, k(\mathcal{Q}_{I_A}, \mathcal{P}), k(\mathcal{P}_{J_B}, \mathcal{Q}), C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2}.$$

- (4) Finally, if

- (a) $\mathcal{O} = \mathcal{O}'$,
 (b) \mathcal{Q}_{I_A} and $w|_{I_A}$ are relatively \mathcal{P} -moderate, and
 (c) \mathcal{P}_{J_B} and $v|_{J_B}$ are relatively \mathcal{Q} -moderate,
 then the following equivalence holds:

$$\begin{aligned} \theta \text{ bounded} &\iff \iota \text{ bounded} \\ &\iff p_1 \leq p_2 \text{ and } K^{(1)} < \infty \text{ and } K^{(2)} < \infty, \end{aligned}$$

where for each $j \in J \setminus J_B$, some $i_j \in I_j$ and for each $i \in I^{(B)}$, some $j_i \in J_i \cap J_B$ is selected, to define

$$\begin{aligned} K^{(1)} &:= \left\| \left(\frac{v_j}{w_{i_j}} \cdot |\det T_{i_j}|^{\frac{1}{p_1} - \frac{1}{p_2} - s_1} \cdot |\det S_j|^{s_1} \right)_{j \in J \setminus J_B} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}, \text{ with } s_1 := \left(\frac{1}{p_2} - \frac{1}{q_1} \right)_+, \\ K^{(2)} &:= \left\| \left(\frac{v_{j_i}}{w_i} \cdot |\det T_i|^{s_2} \cdot |\det S_{j_i}|^{\frac{1}{p_1} - \frac{1}{p_2} - s_2} \right)_{i \in I^{(B)}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}, \text{ with } s_2 := \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm \Delta}} \right)_+, \end{aligned}$$

where $I^{(B)} := \{i \in I \mid J_B \cap J_i \neq \emptyset\}$.

Precisely, given $p_1 \leq p_2$, we have $\|\theta\| \asymp \|\iota\| \asymp K^{(1)} + K^{(2)}$, where the implied constant only depends on d, p_1, p_2, q_1, q_2 and on

$$\begin{aligned} &\mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{Q}}, \varepsilon_{\mathcal{P}}, k(\mathcal{Q}_{I_A}, \mathcal{P}), k(\mathcal{P}_{J_B}, \mathcal{Q}), C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_1}, \\ &C_{\text{mod}}(\mathcal{Q}_{I_A}, \mathcal{P}), C_{\text{mod}}(\mathcal{P}_{J_B}, \mathcal{Q}), C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}, C_{w|_{I_A}, \mathcal{Q}, \mathcal{P}}, C_{v|_{J_B}, \mathcal{P}, \mathcal{Q}}. \end{aligned} \quad \blacktriangleleft$$

Proof. For the whole proof, let $Y := \ell_w^{q_1}(I)$ and $Z := \ell_v^{q_2}(J)$. We will prove each statement individually.

1. This is an immediate consequence of Corollary 5.14, cf. in particular the last statement of that corollary and the ensuing remark. Also note that the triangle constant of $Z = \ell_v^{q_2}(J)$ only depends on q_2 and that Lemma 4.13 can be used to estimate $\|\Gamma_{\mathcal{Q}}\|_{Y \rightarrow Y}$ and $\|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}$ only in terms of $q_1, q_2, \mathcal{Q}, \mathcal{P}, C_{w, \mathcal{Q}}$ and $C_{v, \mathcal{P}}$.

2. First, note that we can apply Lemma 6.1 with $K := \mathcal{O} \cap \mathcal{O}'$. Thus, choose some $\xi \in \mathcal{O} \cap \mathcal{O}'$. Since \mathcal{Q} and \mathcal{P} are open covers of \mathcal{O} and \mathcal{O}' , respectively, there are $i \in I$ and $j \in J$ with

$$\xi \in Q_i \cap P_j \cap K = Q_i^\circ \cap P_j^\circ \cap K^\circ.$$

Furthermore, $\delta_i \in Y = \ell_w^{q_1}(I)$, so that Lemma 6.1 yields $p_1 \leq p_2$, by boundedness of θ .

Next, we apply Theorem 6.13, with $J_0 := J_B$. Note that \mathcal{P}_{J_0} is almost subordinate to \mathcal{Q} by assumption. In particular, $K := \bigcup_{j \in J_0} P_j$ satisfies $K \subset \mathcal{O} \cap \mathcal{O}'$, so that the map $\iota := \theta|_{\mathcal{D}_K}$, with $\mathcal{D}_K = \{f \in C_c^\infty(\mathbb{R}^d) \mid \text{supp } f \subset K\} \subset C_c^\infty(\mathcal{O} \cap \mathcal{O}') \subset \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$, i.e. with $\mathcal{D}_K = \mathcal{D}_K^{\mathcal{Q}, p_1, \ell_w^{q_1}(I)}$ satisfies the assumptions of Theorem 6.13. Since $Z = \ell_v^{q_2}(J)$ satisfies the Fatou property, this implies boundedness of the embedding

$$\eta : \ell_w^{q_1}([\ell^{p_1}(J_i \cap J_B)]_{i \in I}) \hookrightarrow (\ell_v^{q_2}(J_B))_{|\det S_j|^{p_1^{-1} - p_2^{-1}}} = \ell_v^{q_2}_{\left[|\det S_j|^{p_1^{-1} - p_2^{-1}} \cdot v_j\right]_j}(J_B),$$

with $\|\eta\| \lesssim \|\theta\|_{\mathcal{D}_K} \leq \|\theta\|$, where the implied constant only depends on

$$d, p_1, p_2, q_2, k(\mathcal{P}_{J_B}, \mathcal{Q}), \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, C_{v, \mathcal{P}},$$

since the triangle constant for $Z = \ell_v^{q_2}(J)$ only depends on q_2 and since Lemma 4.13 allows us to estimate $\|\Gamma_{\mathcal{P}}\|_{Z \rightarrow Z}$ only in terms of $q_2, \mathcal{P}, C_{v, \mathcal{P}}$.

Finally, an application of Corollary 5.12 (with $r = p_1$, $u \equiv 1$, $u^{(1)} \equiv 1$ and $u^{(2)} \equiv 1$) yields $\|\eta\| \asymp K_{p_1}^{(2,1)}$, where the implied constant only depends on $p_1, q_1, q_2, \mathcal{Q}, k(\mathcal{P}_{J_B}, \mathcal{Q}), C_{w, \mathcal{Q}}$.

To prove the statement involving $K_s^{(1)}$, we invoke Theorem 6.15, with $I_0 = I_A$. To see that it is applicable, note that we have $K := \bigcup_{i \in I_0} Q_i \subset \mathcal{O} \cap \mathcal{O}'$, since \mathcal{Q}_{I_A} is almost subordinate to \mathcal{P} . As above, this implies that the map $\iota := \theta|_{\mathcal{D}_K}$ satisfies the assumptions of Theorem 6.15. Since $Z = \ell_v^{q_2}(J)$ satisfies the Fatou property, this yields boundedness of

$$\eta : \ell_v^{q_1}_{\left(|\det T_i|^{p_2^{-1} - p_1^{-1}} \cdot w_i\right)_i}(I_A) = (Y|_{I_0})_{|\det T_i|^{p_2^{-1} - p_1^{-1}}} \hookrightarrow Z([\ell^s(I_0 \cap I_j)]_{j \in J}) = \ell_v^{q_2}([\ell^s(I_A \cap I_j)]_{j \in J}).$$

Precisely, we get $\|\eta\| \lesssim \|\theta\|_{\mathcal{D}_K} \leq \|\theta\|$, where the implied constant only depends on

$$d, p_1, p_2, q_1, q_2, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{Q}}, k(\mathcal{Q}_{I_A}, \mathcal{P}), C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_2},$$

by the usual arguments involving Lemma 4.13. But by Corollary 5.8 (with $J_0 = J$, $r = s$, $u \equiv 1$ and $(|\det T_i|^{p_2^{-1} - p_1^{-1}} \cdot w_i)_i$ instead of w), we have $K_s^{(1)} \lesssim \|\eta\|$, where the implied constant only depends on $p_2, q_1, q_2, \mathcal{P}, k(\mathcal{Q}_{I_A}, \mathcal{P}), C_{v, \mathcal{P}}$.

3. The proof is similar to the previous case, but using Theorem 6.17 and 6.18 (and the ensuing remarks) instead of Theorems 6.13 and 6.15, respectively.

4. Let us first assume that θ is bounded. As seen above, this yields $p_1 \leq p_2$. We want to show that also $K^{(1)} \lesssim \|\theta\|$ and $K^{(2)} \lesssim \|\theta\|$.

For the first part, we invoke Theorem 6.23, with $J_0 := J \setminus J_B$. Let us verify the prerequisites of that theorem. First of all, we need to verify that \mathcal{Q}_{I_0} is almost subordinate to and relatively moderate with respect to \mathcal{P} , where

$$I_0 = \{i \in I \mid J_i \cap J_0 \neq \emptyset\}.$$

But for $i \in I_0$, there is some $j \in J_0 \cap J_i$, which yields existence of some $\xi \in Q_i \cap P_j \subset \mathcal{O} \cap \mathcal{O}' = A \cup B$. In case of $\xi \in B$, we would have $j \in J_B$, in contradiction to $j \in J_0$. Hence, $\xi \in A$, so that we get $i \in I_A$, since $\xi \in A \cap Q_i$. Thus, we have shown $I_0 \subset I_A$.

Since \mathcal{Q}_{I_A} is almost subordinate to and relatively moderate with respect to \mathcal{P} , so is \mathcal{Q}_{I_0} , with $k(\mathcal{Q}_{I_0}, \mathcal{P}) \leq k(\mathcal{Q}_{I_A}, \mathcal{P})$ and $C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}) \leq C_{\text{mod}}(\mathcal{Q}_{I_A}, \mathcal{P})$. Furthermore, since $w|_{I_A}$ is relatively \mathcal{P} -moderate, so is $w|_{I_0}$, with $C_{w|_{I_0}, \mathcal{Q}, \mathcal{P}} \leq C_{w|_{I_A}, \mathcal{Q}, \mathcal{P}}$. Finally, since \mathcal{P} is tight and since we have $\mathcal{O} = \mathcal{O}'$, assumption (1) of Theorem 6.23 is also satisfied (with $\varepsilon = \varepsilon_{\mathcal{P}}$). All in all, we have verified assumptions (1)–(4) of that theorem. Finally, note that we have $C_{\mathcal{P}, \Psi, p_2} \lesssim C_{\mathcal{P}, \Psi, p_1}$, where the implied constant only depends on d, p_1, \mathcal{P} , cf. Corollary 5.4.

Thus, it remains to verify boundedness of the map ι as defined in Theorem 6.23. But since the set K defined in Theorem 6.23 satisfies $K \subset \mathcal{O} \cap \mathcal{O}'$, we see that this map ι satisfies $\iota = \theta|_{\mathcal{D}_K}$ and hence $\|\iota\| \leq \|\theta\| < \infty$. Hence, Theorem 6.23 yields a constant $C > 0$, depending only on

$$d, p_1, p_2, q_1, q_2, k(\mathcal{Q}_{I_A}, \mathcal{P}), C_{\text{mod}}(\mathcal{Q}_{I_A}, \mathcal{P}), \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{Q}}, \varepsilon_{\mathcal{P}}, C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}, C_{w|_{I_A}, \mathcal{Q}, \mathcal{P}}, C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_1},$$

which satisfies $K^{(1)} \leq C \cdot \|\iota\| \leq C \cdot \|\theta\| < \infty$.

To show $K^{(2)} \lesssim \|\theta\|$, we invoke Theorem 6.21, with $J_0 := J_B$. Let us first verify assumptions (1)–(3) of that theorem. The first two of these are direct consequences of our assumptions and our choice of $J_0 = J_B$. For the last one, we need to find $C_0 > 0$ and $r \in \mathbb{N}_0$ satisfying

$$\lambda(Q_i) \leq C_0 \cdot \lambda \left(\bigcup_{j \in J_0 \cap J_i} P_j^{r*} \right) \quad (7.2)$$

for all $i \in I_0 := \{i \in I \mid J_0 \cap J_i \neq \emptyset\} = I^{(B)}$. To this end, set $k := k(\mathcal{Q}_{I_A}, \mathcal{P})$ and let $i \in I^{(B)}$ be arbitrary. We distinguish two cases:

- Case 1.* We have $i \in I_A$. Because of $i \in I_0 = I^{(B)}$, there is some $\ell \in J_0 \cap J_i$, i.e. with $Q_i \cap P_\ell \neq \emptyset$. Since $i \in I_A$ and because \mathcal{Q}_{I_A} is almost subordinate to \mathcal{P} , Lemma 2.11 yields the inclusion $Q_i \subset P_\ell^{(2k+2)*} \subset \bigcup_{j \in J_0 \cap J_i} P_j^{(2k+2)*}$.
- Case 2.* We have $i \notin I_A$, i.e. $Q_i \cap A = \emptyset$. But we have $Q_i \subset \mathcal{O} = \mathcal{O} \cap \mathcal{O}' = A \cup B$, so that we get $Q_i \subset B$. Now, let $\xi \in Q_i \subset B \subset \mathcal{O} \cap \mathcal{O}' = \mathcal{O}'$ be arbitrary. Then there is some $\ell \in J$ with $\xi \in P_\ell$. Since $\xi \in B$, we get $\xi \in P_\ell \cap B \neq \emptyset$ and hence $\ell \in J_B = J_0$, i.e. $\ell \in J_0 \cap J_i$, which implies $\xi \in P_\ell \subset \bigcup_{j \in J_0 \cap J_i} P_j \subset \bigcup_{j \in J_0 \cap J_i} P_j^{(2k+2)*}$. Since $\xi \in Q_i$ was arbitrary, we get $Q_i \subset \bigcup_{j \in J_0 \cap J_i} P_j^{(2k+2)*}$.

Together, the two cases easily show that we can choose $r = 2k + 2$ and $C_0 = 1$.

We have thus verified all prerequisites formulated in Theorem 6.21 itself; but note that the prerequisites of the theorem also include those of Lemma 6.19, which we verify now. First of all, $\mathcal{P}_{J_0} = \mathcal{P}_{J_B}$ is almost subordinate to \mathcal{Q} , so that it remains to establish existence of a map

$$\iota : \left(\mathcal{D}_K^{\mathcal{Q}, p_1, \ell_w^{q_1}(I)}, \|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})} \right) \rightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$$

with $\langle \iota f, \varphi \rangle = \langle f, \varphi \rangle$ for all $\varphi \in C_c^\infty(\mathcal{O} \cap \mathcal{O}')$ and all $f \in \mathcal{D}_K^{\mathcal{Q}, p_1, \ell_w^{q_1}(I)}$, where

$$K = \bigcup_{i \in I_0} \overline{Q_i^{(2k+3)*}} \subset \mathcal{O} = \mathcal{O} \cap \mathcal{O}'.$$

But given our present assumptions, it is easy to see that $\iota = \theta|_{\mathcal{D}_K^{\mathcal{Q}, p_1, \ell_w^{q_1}(I)}}$ satisfies these properties.

All in all, we can thus apply Theorem 6.21, so that we get a constant $C > 0$, depending only on

$$d, p_1, p_2, q_1, q_2, \mathcal{Q}, \varepsilon_{\mathcal{Q}}, \mathcal{P}, \varepsilon_{\mathcal{P}}, k(\mathcal{Q}_{I_A}, \mathcal{P}), k(\mathcal{P}_{J_B}, \mathcal{Q}), C_{\text{mod}}(\mathcal{P}_{J_B}, \mathcal{Q}), C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}, C_{v|_{J_B}, \mathcal{P}, \mathcal{Q}}, C_{\mathcal{Q}, \Phi, p_1},$$

which satisfies $K^{(2)} \leq C \cdot \|\iota\| \leq C \cdot \|\theta\| < \infty$.

Next, if ι is bounded, the properties of ι from part (1) (together with $\mathcal{O} = \mathcal{O}'$) imply $\iota f = f$ for all $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$, so that the map θ from part (2) satisfies $\theta = \iota|_{C_c^\infty(\mathcal{O})}$. In view of what we just showed, we thus get $p_1 \leq p_2$, $K^{(1)} \lesssim \|\theta\| \leq \|\iota\|$ and $K^{(2)} \lesssim \|\theta\| \leq \|\iota\|$.

It thus remains to show that ι is bounded if $p_1 \leq p_2$, $K^{(1)} < \infty$ and $K^{(2)} < \infty$. To this end, we will show that the assumptions of Corollary 5.14 are satisfied. All of these assumptions are trivially included in those of the present theorem, with the exception of finiteness of the right-hand sides of equations (5.25) and (5.26), which we will verify now.

First, note that the second part of Corollary 5.8 (with $J_0 := J \setminus J_B$, with $I_0 := L := \bigcup_{j \in J \setminus J_B} I_j$ and with $r = p_2^\nabla$ and $u_i := |\det T_i|^{p_1^{-1} - p_2^{-1}}$) yields

$$\begin{aligned} \text{r.h.s. of eq. (5.25)} &= \left\| \left(v_j \cdot \left\| \left(|\det T_i|^{p_1^{-1} - p_2^{-1}} / w_i \right)_{i \in I_j} \right\|_{\ell^{p_2^\nabla \cdot (q_1/p_2^\nabla)'}} \right)_{j \in J \setminus J_B} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ (I_j \subset L = I_0 \text{ for } j \in J_0 = J \setminus J_B) &= \left\| \left(v_j \cdot \left\| (u_i/w_i)_{i \in I_0 \cap I_j} \right\|_{\ell^{p_2^\nabla \cdot (q_1/p_2^\nabla)'}} \right)_{j \in J_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ &\asymp \left\| \left(v_j \cdot \frac{u_{i_j}}{w_{i_j}} \cdot \left[|\det S_j| / |\det T_{i_j}| \right]^{\left(\frac{1}{p_2^\nabla} - \frac{1}{q_1} \right)_+} \right)_{j \in J^{(0)}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ &\stackrel{(*)}{=} K^{(1)} < \infty, \end{aligned} \quad (7.3)$$

with $J^{(0)} := \{j \in J_0 \mid I_0 \cap I_j \neq \emptyset\}$ and with an implied constant only depending on

$$d, p_1, p_2, q_1, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{Q}}, \varepsilon_{\mathcal{P}}, k(\mathcal{Q}_{I_A}, \mathcal{P}), C_{\text{mod}}(\mathcal{Q}_{I_A}, \mathcal{P}), C_{w|_{I_A}, \mathcal{Q}, \mathcal{P}}.$$

The prerequisites for the application of Corollary 5.8 from above are not hard to verify: As seen in Corollary 5.14, we have $I_0 = L \subset I_A$, so that \mathcal{Q}_{I_0} is almost subordinate to and relatively moderate with respect to \mathcal{P} , since \mathcal{Q}_{I_A} is. We also get $C_{\text{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}) \leq C_{\text{mod}}(\mathcal{Q}_{I_A}, \mathcal{P})$ and $k(\mathcal{Q}_{I_0}, \mathcal{P}) \leq k(\mathcal{Q}_{I_A}, \mathcal{P})$. Hence, the only nontrivial point is to establish existence of $C_0 > 0$ and $s \in \mathbb{N}_0$ satisfying

$$\lambda(P_j) \leq C_0 \cdot \lambda \left(\bigcup_{i \in I_0 \cap I_j} Q_i^{s*} \right)$$

for all $j \in J_0$. But since we have $I_0 \cap I_j = L \cap I_j = I_j$ for all $j \in J_0 = J \setminus J_B$, and because of $P_j \subset \mathcal{O}' = \mathcal{O} = \bigcup_{i \in I} Q_i$, we easily see $P_j \subset \bigcup_{i \in I_j} Q_i = \bigcup_{i \in I_0 \cap I_j} Q_i^{0*}$ for all $j \in J_0$, so that we can choose $C_0 = 1$ and $s = 0$.

Finally, to justify the step marked with $(*)$ in estimate (7.3), note that we have $u_{i_j} = |\det T_{i_j}|^{p_1^{-1} - p_2^{-1}}$ and that we have $I_0 \cap I_j = I_j \neq \emptyset$ for all $j \in J_0$, since $\emptyset \neq P_j \subset \bigcup_{i \in I_0 \cap I_j} Q_i$, as we just saw. Hence, $J^{(0)} = J_0 = J \setminus J_B$.

We have thus established finiteness of the right-hand side of equation (5.25).

To establish finiteness of the right-hand side of equation (5.26), we apply the second part of Corollary 5.12, with $r = p_1^\Delta$, $J_0 = J_B$ and

$$u_i^{(1)} := \begin{cases} |\det T_i|^{1-p_1^{-1}}, & \text{if } p_1 < 1, \\ 1, & \text{if } p_1 \geq 1, \end{cases} \quad \text{as well as} \quad u_j^{(2)} := \begin{cases} |\det S_j|^{p_2^{-1}-1}, & \text{if } p_1 < 1, \\ |\det S_j|^{p_2^{-1}-p_1^{-1}}, & \text{if } p_1 \geq 1. \end{cases}$$

All of the prerequisites of Corollary 5.12 are easily seen to be fulfilled, with the possible exception of

$$\lambda(Q_i) \leq C_0 \cdot \lambda \left(\bigcup_{j \in J_0 \cap J_i} P_j^{s*} \right) \quad \forall i \in I^{(0)} = \{i \in I \mid J_0 \cap J_i \neq \emptyset\} = I^{(B)}.$$

But following equation (7.2), we verified exactly this estimate, for $C_0 = 1$ and $s = 2k + 2$, for $k = k(\mathcal{Q}_{I_A}, \mathcal{P})$. All in all, Corollary 5.12 thus yields

$$\begin{aligned} \text{r.h.s. of eq. (5.26)} &= \left\| \left(w_i^{-1} \cdot \left\| (v_j/u_{i,j})_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2 \cdot (p_1^\Delta/q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ &\asymp \left\| \left(w_i^{-1} \cdot \frac{v_{j_i}}{u_{i,j_i}} \cdot \left[|\det T_i| / |\det S_{j_i}| \right]^{\left(\frac{1}{q_2} - \frac{1}{p_1^\Delta} \right)_+} \right)_{i \in I^{(B)}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \end{aligned}$$

$$(\text{cf. Remark 5.13}) = K^{(2)} < \infty,$$

where the implied constant only depends on

$$d, p_1, p_2, q_2, \mathcal{Q}, \mathcal{P}, \varepsilon_{\mathcal{Q}}, \varepsilon_{\mathcal{P}}, k(\mathcal{Q}_{I_A}, \mathcal{P}), k(\mathcal{P}_{J_B}, \mathcal{Q}), C_{\text{mod}}(\mathcal{P}_{J_B}, \mathcal{Q}), C_{v|_{J_B}, \mathcal{P}, \mathcal{Q}}.$$

Finally, we can invoke Corollary 5.14 to conclude that ι is bounded, with

$$\|\iota\| \leq C \cdot [(\text{r.h.s. of eq. (5.25)}) + (\text{r.h.s. of eq. (5.26)})],$$

for some constant

$$C = C(d, p_1, p_2, q_1, q_2, \mathcal{Q}, \mathcal{P}, k(\mathcal{P}_{J_B}, \mathcal{Q}), k(\mathcal{Q}_{I_A}, \mathcal{P}), C_{\mathcal{Q}, \Phi, p_1}, C_{\mathcal{P}, \Psi, p_1}, C_{w, \mathcal{Q}}, C_{v, \mathcal{P}}),$$

thereby completing the proof. \square

Before we close this section, we give a quick user's guide which describes a convenient workflow for deciding the existence of an embedding between decomposition spaces. Concrete applications of (essentially) this workflow are given in Section 9. The typical approach is as follows:

- (1) Determine the coverings \mathcal{Q}, \mathcal{P} (and the remaining parameters p_1, p_2, q_1, q_2, w, v) which define the decomposition spaces for which an embedding $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$ is desired.
- (2) If $p_1 > p_2$, the embedding does not exist. Thus, assume in the following that $p_1 \leq p_2$.
- (3) Determine whether \mathcal{P} is almost subordinate to \mathcal{Q} or vice versa (or neither). Probably the most convenient way to do this is as follows:
 - (a) Determine upper- and lower bounds (in terms of set-inclusion) for the intersection sets

$$I_j := \{i \in I \mid Q_i \cap P_j \neq \emptyset\} \quad \text{and} \quad J_i := \{j \in J \mid P_j \cap Q_i \neq \emptyset\}.$$

This amounts to finding conditions on i, j which ensure/prevent $Q_i \cap P_j \neq \emptyset$. This step is the (only) one in which the *geometry* of the two coverings plays a significant role.

- (b) If we have $\sup_{j \in J} |I_j| < \infty$, then \mathcal{P} is **weakly subordinate** to \mathcal{Q} . If \mathcal{P} consists of path-connected sets and if \mathcal{Q} is an open covering, then this implies that \mathcal{P} is almost subordinate to \mathcal{Q} , cf. Lemma 2.12.

The same of course holds if \mathcal{Q} and \mathcal{P} are exchanged everywhere and if the condition $\sup_{j \in J} |I_j| < \infty$ is replaced by $\sup_{i \in I} |J_i| < \infty$.

- (4) A quick check: If $p_1 = p_2$, try to prove/disprove that $v_j \lesssim w_i$ if $Q_i \cap P_j \neq \emptyset$. If this fails, the embedding does *not* exist.
- (5) Determine whether \mathcal{P} and v are relatively moderate with respect to \mathcal{Q} (or vice versa).

This amounts to showing that any two sets P_j, P_ℓ with $j, \ell \in J_i$ have essentially the same measure, $\lambda(P_j) \asymp \lambda(P_\ell)$ and that $v_j \asymp v_\ell$ for $j, \ell \in J_i$. Of course, this depends on having good upper bounds for the sets $(J_i)_{i \in I}$. Usually, this step will yield sequences $\gamma_i^{(1)}, \gamma_i^{(2)}$ satisfying

$$\lambda(P_j) \asymp \gamma_i^{(1)} \quad \text{and} \quad v_j \asymp \gamma_i^{(2)} \quad \text{for } j \in J_i \text{ and } i \in I. \quad (7.4)$$

- (6) Choose the right criterion to use:

- (a) If \mathcal{Q} is almost subordinate to \mathcal{P} : Consider Theorem 7.4. Specifically,
 - (i) If \mathcal{Q} and w are relatively \mathcal{P} -moderate: Use part (4) of Theorem 7.4. This requires only to decide finiteness of a *single norm*, which can usually be easily computed (using the equivalents of the weights $\gamma^{(1)}$ and $\gamma^{(2)}$ from equation (7.4)). Furthermore, it yields a *complete characterization* of the existence of the embedding, i.e. a *yes/no answer*.
 - (ii) If \mathcal{Q} or w are *not* relatively \mathcal{P} -moderate: Consider the expression

$$K(r) := \left\| \left(v_j \cdot \left\| \left(|\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}} / w_i \right)_{i \in I_j} \right\|_{\ell^{r \cdot (q_1/r)'} } \right)_{j \in J} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}$$

from Theorem 7.4 (equation (7.1)) for general $r \in (0, \infty]$. More precisely, determine for which values of r , the expression K_r is finite. For this, it could be helpful to note $\frac{1}{r \cdot (q_1/r)'} = \left(\frac{1}{r} - \frac{1}{q_1} \right)_+$. In particular, $r \mapsto K_r$ is non-increasing.

- (A) If $K(p_2^\vee) < \infty$, the embedding exists.
- (B) If $K(p_2) = \infty$ or if $p_2 = \infty$ and $K(p_2^\vee) = \infty$, the embedding does *not* exist.
- (C) If $p_1 = p_2$ and $K(2) = \infty$, the embedding does *not* exist.
- (D) If none of the cases above applies, the (convenient) criteria given in this paper are inconclusive.

- (b) If \mathcal{P} is almost subordinate to \mathcal{Q} : Consider Theorem 7.2. Specifically,
- (i) If \mathcal{P} and v are relatively \mathcal{Q} -moderate: Use part (4) of Theorem 7.2. This requires only to decide finiteness of a *single norm*, which can usually be easily computed (using the weights $\gamma^{(1)}$ and $\gamma^{(2)}$ from equation (7.4)). Furthermore, it yields a *complete characterization* of the existence of the embedding, i.e. a *yes/no answer*.
 - (ii) If \mathcal{P} or v are not relatively \mathcal{Q} -moderate: Consider the expression

$$K(r, \alpha, \beta) := \left\| \left(w_i^{-1} \cdot |\det T_i|^\alpha \cdot \left\| \left(|\det S_j|^\beta \cdot v_j \right)_{j \in J_i} \right\|_{\ell^{q_2 \cdot (r/q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}$$

for general $r \in (0, \infty]$ and $\alpha, \beta \in \mathbb{R}$. Then,

- (A) If $p_1 \geq 1$ and if $K\left(p_1^\Delta, 0, \frac{1}{p_1} - \frac{1}{p_2}\right) < \infty$, the embedding exists.
 - (B) If $p_1 < 1$ and if $K\left(p_1^\Delta, \frac{1}{p_1} - 1, 1 - \frac{1}{p_2}\right) < \infty$, the embedding exists.
 - (C) If $K\left(p_1, 0, \frac{1}{p_1} - \frac{1}{p_2}\right) = \infty$, the embedding does *not* exist.
 - (D) If $p_1 = p_2$ and $K(2, 0, 0) = \infty$, the embedding does *not* exist.
 - (E) If none of the cases above applies, the (convenient) criteria given in this paper are inconclusive.
- (c) If neither \mathcal{Q} is almost subordinate to \mathcal{P} , nor vice versa.
- (i) Try to find sets A, B satisfying $A \cup B = \mathcal{O} \cap \mathcal{O}'$ and such that \mathcal{Q} is “smaller than” \mathcal{P} “near” A and vice versa “near” B . Precisely, such that if

$$I_A := \{i \in I \mid Q_i \cap A \neq \emptyset\} \quad \text{and} \quad J_B := \{j \in J \mid P_j \cap B \neq \emptyset\},$$

then $\mathcal{Q}_{I_A} = (Q_i)_{i \in I_A}$ is almost subordinate to \mathcal{P} and $\mathcal{P}_{J_B} = (P_j)_{j \in J_B}$ is almost subordinate to \mathcal{Q} .

If such sets can be found, use Theorem 7.5. A more detailed explanation is outside of the scope of this user’s guide. See the mentioned theorem for more details.

- (ii) If a sufficient criterion is desired:
 - (A) Try to find a covering \mathcal{R} such that \mathcal{Q} and \mathcal{P} are both almost subordinate to \mathcal{R} or such that \mathcal{R} is almost subordinate to both \mathcal{Q} and \mathcal{P} . Then, use the preceding criteria to establish a chain of embeddings

$$\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{R}, L^{p_3}, \ell_u^{q_3}) \hookrightarrow \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$$

for suitably selected parameters p_3, q_3 and a suitable weight u .

- (B) Try to apply the (very general) Theorem 5.6. This theorem does not require subordinateness of any kind, but its prerequisites are quite involved and fall outside the scope of this user’s guide.
- (iii) If a necessary criterion is desired: There is no convenient criterion in this paper which applies in this generality. However, reading the statements and proofs of our various necessary conditions might yield inspiration for finding a “custom-tailored” necessary criterion. The interested reader might want to study the proof of equation (9.23) in the proof of Theorem 9.21 as an example of such a criterion.

8. DECOMPOSITION SPACES AS SPACES OF TEMPERED DISTRIBUTIONS

In this section we study embeddings of decomposition spaces into the space of tempered distributions. This question is nontrivial, since the (Fourier-side) decomposition spaces in this paper are defined as subspaces of $\mathcal{D}'(\mathcal{O})$ instead of $\mathcal{S}'(\mathbb{R}^d)$.

As noted in Section 3—in particular in Remark 3.13 and Example 3.22—this has two main reasons:

- (1) We want to allow the case $\mathcal{O} \subsetneq \mathbb{R}^d$. In this case, one would have to factor out a certain subspace of $\mathcal{S}'(\mathbb{R}^d)$ to obtain a positive definite quasi-norm $\|\cdot\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)}$.
- (2) Even for $\mathcal{O} = \mathbb{R}^d$, using $\mathcal{S}'(\mathbb{R}^d)$ as the reservoir can lead to incomplete spaces $\mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^p, Y)$, even in case of $Y = \ell_u^1$ with a \mathcal{Q} -moderate weight u .

The second reason can be seen as just a technical nuisance in most cases. That is why we now provide a readily verifiable sufficient criterion which ensures that each element $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y) \subset \mathcal{D}'(\mathcal{O})$ of a (Fourier-side) decomposition space has an extension to a tempered distribution. In case of $\mathcal{O} = \mathbb{R}^d$, this also implies that every element $f \in \mathcal{D}(\mathcal{Q}, L^p, Y)$ of the space-side decomposition space admits an extension $\mathcal{F}^{-1}\hat{f} \in \mathcal{S}'(\mathbb{R}^d)$, since $\mathcal{S}'(\mathbb{R}^d)$ is invariant under the Fourier transform and because of $\hat{f} \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$. This implies that the two spaces $\mathcal{D}(\mathcal{Q}, L^p, Y)$ and $\mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^p, Y)$ (as defined in Remark 3.13) coincide—up to obvious identifications.

Our criterion can be most conveniently formulated using the notions of so-called **regular partitions of unity** and **regular coverings**. These terms were first used informally by Borup and Nielsen[4] and formally introduced in [24, Definition 2.4].

Definition 8.1. Let $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ be a semi-structured covering of an open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ and let $\Phi = (\varphi_i)_{i \in I}$ be a smooth partition of unity subordinate to \mathcal{O} . For $i \in I$, define the normalized version of φ_i by

$$\varphi_i^\# : \mathbb{R}^d \rightarrow \mathbb{C}, \xi \mapsto \varphi_i(T_i \xi + b_i).$$

We say that Φ is a **regular partition of unity** subordinate to \mathcal{Q} if $\varphi_i \in C_c^\infty(\mathcal{O})$ with $\varphi_i \equiv 0$ on $\mathcal{O} \setminus Q_i$ for all $i \in I$ and if additionally

$$C_{\Phi, \alpha} := \sup_{i \in I} \|\partial^\alpha \varphi_i^\#\|_{\sup}$$

is finite for all $\alpha \in \mathbb{N}_0^d$.

\mathcal{Q} is called a **regular covering** of \mathcal{O} if there is a regular partition of unity Φ subordinate to \mathcal{Q} . ◀

Although the notion of a regular covering seems somewhat restrictive, every “reasonable” covering turns out to be regular. For readers who want to recall the notion of an **almost structured covering**, we refer to Definition 2.5 from above.

Theorem 8.2. (cf. [24, Corollary 2.7 and Theorem 2.8]; see [4, Proposition 1] for a similar statement)

Every almost structured covering is regular. In particular, every structured covering is regular.

Finally, every regular partition of unity is an L^p -BAPU for all $p \in (0, \infty]$. ◀

Using the notion of regular partitions of unity, we can now state our criterion for embeddings into the space of tempered distributions. One can probably tweak some of the parameters to obtain similar, but different sufficient criteria. But for our purposes, the present version suffices.

Theorem 8.3. Assume that $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ is a regular covering of the open set $\emptyset \neq \mathcal{O} \subset \mathbb{R}^d$ and let $\Phi = (\varphi_i)_{i \in I}$ be a regular partition of unity for \mathcal{Q} . Let $Y \leq \mathbb{C}^I$ be \mathcal{Q} -regular and let $p \in (0, \infty]$.

For each $N \in \mathbb{N}_0$, define $w^{(N)} = (w_i^{(N)})_{i \in I}$ by

$$w_i^{(N)} := |\det T_i|^{1/p} \cdot \max \left\{ 1, \|T_i^{-1}\|^{d+1} \right\} \cdot \left[\inf_{\xi \in Q_i^*} (1 + |\xi|) \right]^{-N}.$$

If there is some subset $I_0 \subset I$ and some $N \in \mathbb{N}_0$ for which we have

$$w^{(N)} \cdot \mathbf{1}_{I_0} \cdot c \in \ell^1(I) \quad \forall c = (c_i)_{i \in I} \in Y, \quad (8.1)$$

then

$$\Phi : \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y) \rightarrow \mathcal{S}'(\mathbb{R}^d), f \mapsto \left[g \mapsto \sum_{i \in I_0} \langle \varphi_i f, g \rangle_{\mathcal{S}', \mathcal{S}} \right]$$

is well-defined and continuous with respect to the weak-*topology on $\mathcal{S}'(\mathbb{R}^d)$. The series defining $\langle \Phi f, g \rangle_{\mathcal{S}', \mathcal{S}}$ converges absolutely for every $g \in \mathcal{S}(\mathbb{R}^d)$.

Finally, for $I_0 = I$, the tempered distribution $\Phi f \in \mathcal{S}'(\mathbb{R}^d)$ is an extension of $f : C_c^\infty(\mathcal{O}) \rightarrow \mathbb{C}$ to $\mathcal{S}(\mathbb{R}^d)$ for each $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$. ◀

Remark. In case of $Y = \ell_u^q(I)$, it is not hard to see that the assumption (8.1) regarding the weight $w^{(N)}$ is equivalent to

$$w^{(N)} \in \ell_{1/u}^{q'}(I_0). \quad \blacklozenge$$

Proof. We start the proof with some preliminary observations, estimates and definitions.

By assumption, the linear map

$$\Lambda : Y \rightarrow \ell^1(I), c \mapsto w^{(N)} \cdot \mathbf{1}_{I_0} \cdot c$$

is well-defined. Furthermore, since convergence in Y and in $\ell^1(I)$ yields convergence in each coordinate (cf. Lemma 3.8), it is easy to see that Λ has a closed graph. Finally, Y is a Quasi-Banach space and thus¹⁹ an F -space in the terminology of [16, Section 1.8]. Hence, the closed graph theorem (cf. [16, Theorem 2.15]) shows that Λ is continuous and hence bounded, i.e.

$$\|(w_i^{(N)} \cdot c_i)_{i \in I_0}\|_{\ell^1(I_0)} = \|w^{(N)} \cdot \mathbf{1}_{I_0} \cdot c\|_{\ell^1(I)} \leq C_1 \cdot \|c\|_Y \quad \forall c \in Y \quad (8.2)$$

for some finite constant $C_1 > 0$.

For brevity, let us set

$$S_i : \mathbb{R}^d \rightarrow \mathbb{R}^d, \xi \mapsto T_i \xi + b_i \quad \text{and} \quad c_i := \inf_{y \in Q_i^*} (1 + |y|)$$

for $i \in I$. Furthermore, let

$$\|g\|_{k,N} := \max_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \leq N}} \sup_{x \in \mathbb{R}^d} \left[(1 + |x|)^k \cdot |(\partial^\alpha g)(x)| \right]$$

for $g \in \mathcal{S}(\mathbb{R}^d)$ and $k, N \in \mathbb{N}_0$.

Lemma 5.3 furnishes a constant $C_2 = C_2(d, p, \mathcal{Q}) > 0$ such that

$$\|\mathcal{F}^{-1}(\varphi_i f)\|_{L^\infty} \leq C_2 \cdot |\det T_i|^{1/p} \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p} \quad (8.3)$$

holds for all $i \in I$ and all $f \in \mathcal{D}'(\mathcal{O}) \supset \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$.

For $\ell \in \underline{d}$, $k \in \mathbb{N}_0$, and $j \in I$, the chain rule²⁰ implies (with $\varphi_j^\# := \varphi_j \circ S_j$ as in Definition 8.1) that

$$\begin{aligned} |(\partial_\ell^k \varphi_j)(\xi)| &= \left| \left(\partial_\ell^k \left(\varphi_j^\# \circ S_j^{-1} \right) \right) (\xi) \right| \\ &= \left| \left(\partial_\ell^k \left[\eta \mapsto \varphi_j^\# (T_j^{-1}(\eta - b_j)) \right] \right) (\xi) \right| \\ &= \left| \left(\partial_\ell^k \left[\varphi_j^\# \circ T_j^{-1} \right] \right) (\xi - b_j) \right| \\ &\leq C_3^{(k)} \cdot \|T_j^{-1}\|^k \cdot \max_{|\alpha| \leq k} \|\partial^\alpha \varphi_j^\#\|_{\sup} \\ &\leq C_3^{(k)} \cdot \|T_j^{-1}\|^k \cdot \max_{|\alpha| \leq k} C_{\Phi, \alpha} \\ &\leq C_4^{(k)} \cdot \|T_j^{-1}\|^k \end{aligned}$$

for suitable constants $C_3^{(k)}, C_4^{(k)}$ depending only on $d \in \mathbb{N}$, on $k \in \mathbb{N}_0$ and on the constants $(C_{\Phi, \beta})_{\beta \in \mathbb{N}_0^d}$ from the definition of a regular partition of unity (cf. Definition 8.1).

If we additionally assume $j \in i^*$ for some $i \in I$, we get

$$\|T_j^{-1}\| = \|T_j^{-1} T_i T_i^{-1}\| \leq \|T_j^{-1} T_i\| \cdot \|T_i^{-1}\| \leq C_{\mathcal{Q}} \cdot \|T_i^{-1}\|,$$

¹⁹To see that every Quasi-Banach space is an F -space, note that the Aoki-Rolewicz Theorem (cf. [5, Chapter 2, Theorem 1.1]) shows that Y admits an equivalent r -norm $\|\cdot\|$ for some $r \in (0, 1]$. Then, the metric $d(x, y) := \|x - y\|^r$ is a complete, translation-invariant metric on Y which induces the same topology as the original quasi-norm on Y . Hence, Y is an F -space.

²⁰See [24, Lemma 2.6] for a more detailed explanation of how the chain rule is applied here.

which leads to

$$\begin{aligned}
 |(\partial_\ell^k \varphi_i^*)(\xi)| &\leq \sum_{j \in i^*} |(\partial_\ell^k \varphi_j)(\xi)| \\
 &\leq C_4^{(k)} \cdot \sum_{j \in i^*} \|T_j^{-1}\|^k \\
 &\leq C_4^{(k)} N_{\mathcal{Q}} C_{\mathcal{Q}}^k \cdot \|T_i^{-1}\|^k \\
 &=: C_5^{(k)} \cdot \|T_i^{-1}\|^k.
 \end{aligned}$$

Because of $\varphi_i^* \equiv 0$ on $\mathbb{R}^d \setminus Q_i^*$, we arrive at

$$|(\partial_\ell^k \varphi_i^*)(\xi)| \leq C_5^{(k)} \cdot \|T_i^{-1}\|^k \cdot \mathbf{1}_{\overline{Q_i^*}}(\xi) \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } i \in I. \quad (8.4)$$

Now, let $g \in \mathcal{S}(\mathbb{R}^d)$ and $k \in \mathbb{N}_0$ be arbitrary. Using Leibniz's rule we derive

$$|(\partial_\ell^k [\varphi_i^* \cdot g])(\xi)| \leq \sum_{j=0}^k \left[\binom{k}{j} \cdot |(\partial_\ell^{k-j} \varphi_i^*)(\xi)| \cdot |(\partial_\ell^j g)(\xi)| \right].$$

But using $c_i = \inf_{y \in Q_i^*} (1 + |y|) = \min_{y \in \overline{Q_i^*}} (1 + |y|)$ and estimate (8.4), we see

$$\begin{aligned}
 (1 + |\xi|)^{-N} |(\partial_\ell^{k-j} \varphi_i^*)(\xi)| &\leq C_5^{(k-j)} \cdot \|T_i^{-1}\|^{k-j} \cdot \mathbf{1}_{\overline{Q_i^*}}(\xi) \cdot (1 + |\xi|)^{-N} \\
 &\leq C_5^{(k-j)} \cdot \|T_i^{-1}\|^{k-j} \cdot \left[\inf_{y \in Q_i^*} (1 + |y|) \right]^{-N} \\
 &\leq C_5^{(k-j)} \cdot c_i^{-N} \cdot \max \left\{ 1, \|T_i^{-1}\|^k \right\}
 \end{aligned}$$

for all $\xi \in \mathbb{R}^d$. Furthermore,

$$(1 + |\xi|)^{N+d+1} \cdot |(\partial_\ell^j g)(\xi)| \leq \|g\|_{N+d+1,k}$$

also holds for all $\xi \in \mathbb{R}^d$ and $j \in \{0, \dots, k\}$.

By setting $C_6^{(k)} := \sum_{j=0}^k \binom{k}{j} C_5^{(k-j)}$, a combination of the three estimates above shows

$$|(\partial_\ell^k [\varphi_i^* \cdot g])(\xi)| \leq C_6^{(k)} \|g\|_{N+d+1,k} \cdot c_i^{-N} \cdot \max \left\{ 1, \|T_i^{-1}\|^k \right\} \cdot (1 + |\xi|)^{-d-1}$$

for arbitrary $\xi \in \mathbb{R}^d$, $\ell \in \underline{d}$ and $k \in \mathbb{N}_0$. For $C_7^{(k)} := C_6^{(k)} \cdot \left\| (1 + |\xi|)^{-d-1} \right\|_{L^1}$, this implies

$$\left\| \partial_\ell^k (\varphi_i^* \cdot g) \right\|_{L^1} \leq C_7^{(k)} \cdot \|g\|_{N+d+1,k} \cdot c_i^{-N} \cdot \max \left\{ 1, \|T_i^{-1}\|^k \right\} \quad \forall i \in I. \quad (8.5)$$

Next, it is not hard to see that we have

$$(1 + |\xi|)^{d+1} \leq M_1 \cdot \left(1 + \sum_{\ell=1}^d |\xi_\ell|^{d+1} \right)$$

for all $\xi \in \mathbb{R}^d$ for a suitable constant $M_1 > 0$ which only depends on $d \in \mathbb{N}$. Employing the well-known identity (see e.g. [8, Theorem 8.22(e)])

$$\xi^\alpha \cdot \widehat{g}(\xi) = \left(\frac{1}{2\pi i} \right)^{|\alpha|} \cdot \widehat{\partial^\alpha g}(\xi) \quad \forall \xi \in \mathbb{R}^d$$

for $g \in \mathcal{S}(\mathbb{R}^d)$ and setting $M_2 := M_1 \cdot \left\| (1 + |\xi|)^{-d-1} \right\|_{L^1}$, we thus derive

$$\begin{aligned}
 \left\| \widehat{\varphi_i^* \cdot g} \right\|_{L^1} &= \left\| (1 + |\xi|)^{-d-1} \cdot (1 + |\xi|)^{d+1} \cdot \widehat{\varphi_i^* \cdot g} \right\|_{L^1} \\
 &\leq M_1 \cdot \left\| (1 + |\xi|)^{-d-1} \right\|_{L^1} \cdot \left\| \left(1 + \sum_{\ell=1}^d |\xi_\ell|^{d+1} \right) \cdot \widehat{\varphi_i^* \cdot g} \right\|_{\sup} \\
 &\leq M_2 \cdot \left[\left\| \widehat{\varphi_i^* \cdot g} \right\|_{\sup} + \sum_{\ell=1}^d \left\| \xi_\ell^{d+1} \cdot \widehat{\varphi_i^* \cdot g} \right\|_{\sup} \right] \\
 &= M_2 \cdot \left[\left\| \widehat{\varphi_i^* \cdot g} \right\|_{\sup} + \sum_{\ell=1}^d \left| (2\pi i)^{-(d+1)} \right| \left\| \mathcal{F} [\partial_\ell^{d+1} (\varphi_i^* \cdot g)] \right\|_{\sup} \right] \\
 &\leq M_2 \cdot \left[\left\| \varphi_i^* \cdot g \right\|_{L^1} + \sum_{\ell=1}^d \left\| \partial_\ell^{d+1} (\varphi_i^* \cdot g) \right\|_{L^1} \right] \\
 &\leq (d+1) M_2 \cdot \max \left\{ C_7^{(0)}, C_7^{(d+1)} \right\} \cdot \|g\|_{N+d+1, d+1} \cdot c_i^{-N} \cdot \max \left\{ 1, \|T_i^{-1}\|^{d+1} \right\},
 \end{aligned} \tag{8.6}$$

where the last step used estimate (8.5) and that the quantity

$$\|g\|_{N+d+1, k} \cdot \max \left\{ 1, \|T_i^{-1}\|^k \right\}$$

is nondecreasing as a function of k (because both factors are). The step before used boundedness of $\mathcal{F} : L^1(\mathbb{R}^d) \rightarrow C_0(\mathbb{R}^d)$.

Setting $M_3 := (d+1) M_2 \cdot \max \left\{ C_7^{(0)}, C_7^{(d+1)} \right\}$ and $v_i^{(N)} := c_i^{-N} \cdot \max \left\{ 1, \|T_i^{-1}\|^{d+1} \right\}$ for each $i \in I$, we have thus shown

$$\left\| \widehat{\varphi_i^* \cdot g} \right\|_{L^1} \leq M_3 \cdot \|g\|_{N+d+1, d+1} \cdot v_i^{(N)} \quad \forall i \in I. \tag{8.7}$$

Next, recall from Lemma 2.4 that $\varphi_i^* \equiv 1$ holds on Q_i . Consequently, $\varphi_i^* \varphi_i = \varphi_i$. Furthermore, note for $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y) \leq \mathcal{D}'(\mathcal{O})$ that $\varphi_i f$ is a distribution on \mathbb{R}^d with compact support (since $\varphi_i \in C_c^\infty(\mathcal{O})$) and hence $\varphi_i f \in \mathcal{S}'(\mathbb{R}^d)$. Hence, we arrive at

$$\begin{aligned}
 \sum_{i \in I_0} \left| \langle \varphi_i f, g \rangle_{\mathcal{S}', \mathcal{S}} \right| &= \sum_{i \in I_0} \left| \langle \varphi_i f, \varphi_i^* g \rangle_{\mathcal{S}', \mathcal{S}} \right| \\
 &= \sum_{i \in I_0} \left| \left\langle \mathcal{F}^{-1} [\varphi_i f], \widehat{\varphi_i^* g} \right\rangle_{\mathcal{S}', \mathcal{S}} \right| \\
 &\leq \sum_{i \in I_0} \left[\left\| \mathcal{F}^{-1} [\varphi_i f] \right\|_{L^\infty} \cdot \left\| \widehat{\varphi_i^* g} \right\|_{L^1} \right] \\
 (\text{eqs. (8.3) and (8.7)}) &\leq C_2 M_3 \cdot \|g\|_{N+d+1, d+1} \cdot \sum_{i \in I_0} \left[\left\| \mathcal{F}^{-1} [\varphi_i f] \right\|_{L^p} \cdot |\det T_i|^{\frac{1}{p}} v_i^{(N)} \right] \\
 &= C_2 M_3 \cdot \|g\|_{N+d+1, d+1} \cdot \left\| \left(\left\| \mathcal{F}^{-1} [\varphi_i f] \right\|_{L^p} \cdot w_i^{(N)} \right)_{i \in I_0} \right\|_{\ell^1} \\
 (\text{eq. (8.2)}) &\leq C_1 C_2 M_3 \cdot \|g\|_{N+d+1, d+1} \cdot \left\| \left(\left\| \mathcal{F}^{-1} [\varphi_i f] \right\|_{L^p} \right)_{i \in I} \right\|_Y \\
 &= C_1 C_2 M_3 \cdot \|g\|_{N+d+1, d+1} \cdot \|f\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}, L^p, Y)} < \infty.
 \end{aligned}$$

This estimate proves that $\Phi f \in \mathcal{S}'(\mathbb{R}^d)$ is well-defined with absolute convergence of the series defining $\langle \Phi f, g \rangle_{\mathcal{S}', \mathcal{S}}$ for every $g \in \mathcal{S}(\mathbb{R}^d)$.

Additionally, we see $\langle \Phi f, g \rangle_{\mathcal{S}', \mathcal{S}} \rightarrow 0$ for $f \rightarrow 0$ in $\mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^p, Y)$, for arbitrary $g \in \mathcal{S}(\mathbb{R}^d)$, so that Φ is continuous, as claimed.

Finally, Lemma 2.4 shows that $(\varphi_i)_{i \in I}$ is a locally finite partition of unity on \mathcal{O} , so that we get $g = \sum_{i \in I} \varphi_i g$ for every $g \in C_c^\infty(\mathcal{O})$, where only finitely many terms of the sum do not vanish. In case

of $I_0 = I$, this implies

$$\langle \Phi f, g \rangle_{S', S} = \sum_{i \in I} \langle \varphi_i f, g \rangle_{S', S} = \sum_{i \in I} \langle f, \varphi_i g \rangle_{\mathcal{D}', \mathcal{D}} = \left\langle f, \sum_{i \in I} \varphi_i g \right\rangle_{\mathcal{D}', \mathcal{D}} = \langle f, g \rangle_{\mathcal{D}', \mathcal{D}},$$

for all $g \in C_c^\infty(\mathcal{O})$, so that Φf is indeed an extension of f . \square

9. APPLICATIONS

In this section, we demonstrate the power and ease-of-use of our results by considering two classes of classical spaces, for which our general results extend the state of the art. First, we *completely characterize* the existence of embeddings $M_{s_1, \alpha_1}^{p_1, q_1}(\mathbb{R}^d) \hookrightarrow M_{s_2, \alpha_2}^{p_2, q_2}(\mathbb{R}^d)$, for arbitrary “smoothness parameters” $s_1, s_2 \in \mathbb{R}$, integrability exponents $p_1, p_2, q_1, q_2 \in (0, \infty]$ and $\alpha_1, \alpha_2 \in [0, 1]$. The best known result of which I am aware is the characterization of these embeddings by Han and Wang[13]; but they only allow $(p_1, q_1) \neq (p_2, q_2)$ in case of $\alpha_1 = \alpha_2$, whereas they assume $(p_1, q_1) = (p_2, q_2)$ in case of $\alpha_1 \neq \alpha_2$. Our results need no such restriction.

A curious feature of our treatment is that most of the work is needed to obtain a proper understanding of the relation between the coverings $\mathcal{Q}^{(\alpha)}$ which are used to define the α -modulation spaces. Once this understanding is obtained, the actual embedding statements can be derived with ease. In many other treatments, the properties of the coverings $\mathcal{Q}^{(\alpha)}$ which we formally derive here are essentially taken for granted.

Secondly, we consider the question of embeddings between homogeneous and inhomogeneous Besov spaces.

We remark that a huge number of other examples—including shearlet smoothness spaces and a large class of wavelet-type coorbit spaces—can be handled using the framework developed in this paper. But since the present paper already has a considerable length, we postpone these applications to a later contribution. The interested reader can find many of these applications in my PhD thesis [23]. The examples considered in this section are also considered there.

9.1. Embeddings of α -modulation spaces. We begin our study of α -modulation spaces by describing the construction of the associated covering $\mathcal{Q}^{(\alpha)}$. Our treatment is based on that of Borup and Nielsen[2].

Theorem 9.1. (cf. [2, Theorem 2.6]) Let $d \in \mathbb{N}$ and $0 \leq \alpha < 1$ be arbitrary and set $\alpha_0 := \frac{\alpha}{1-\alpha}$. Then there is a constant $r_1 = r_1(d, \alpha) > 0$ such that the family

$$\mathcal{Q}^{(\alpha)} := \mathcal{Q}_r^{(\alpha)} := \left(Q_{r, k}^{(\alpha)} \right)_{k \in \mathbb{Z}^d \setminus \{0\}} := \left(B_{r|k|^{\alpha_0}}(|k|^{\alpha_0} k) \right)_{k \in \mathbb{Z}^d \setminus \{0\}}$$

is an admissible covering of \mathbb{R}^d for every $r > r_1$. \blacktriangleleft

Our plan is to examine the (relative) geometry of the coverings $\mathcal{Q}^{(\alpha)}$ by

- (1) showing that the covering $\mathcal{Q}^{(\alpha)}$ is a *structured* admissible covering of \mathbb{R}^d ,
- (2) showing that the weight $\mathbb{Z}^d \setminus \{0\} \rightarrow (0, \infty), k \mapsto \langle k \rangle := \sqrt{1 + |k|^2}$ is moderate with respect to $\mathcal{Q}^{(\alpha)}$,
- (3) showing that $\mathcal{Q}^{(\alpha)}$ is almost subordinate to and relatively moderate with respect to $\mathcal{Q}^{(\beta)}$ for $\alpha \leq \beta$.

We begin with establishing the second point.

Lemma 9.2. Let $d \in \mathbb{N}$, $0 \leq \alpha < 1$ and $r > 0$ so that $\mathcal{Q}^{(\alpha)} = \mathcal{Q}_r^{(\alpha)}$ is an admissible covering of \mathbb{R}^d . We then have

$$\langle \xi \rangle \asymp \langle k \rangle^{\frac{1}{1-\alpha}} \quad \text{for all} \quad k \in \mathbb{Z}^d \setminus \{0\} \quad \text{and} \quad \xi \in Q_{r, k}^{(\alpha)},$$

where the implied constant only depends on r, α .

In particular, the weight $w^{(\gamma)} = (\langle k \rangle^\gamma)_{k \in \mathbb{Z}^d \setminus \{0\}}$ is moderate with respect to $\mathcal{Q}_r^{(\alpha)}$ for all $\gamma \in \mathbb{R}$. \blacktriangleleft

Proof. Define $\alpha_0 := \frac{\alpha}{1-\alpha} \in [0, \infty)$. For $k \in \mathbb{Z}^d \setminus \{0\}$ and arbitrary $\xi \in Q_{r,k}^{(\alpha)} = B_{r|k|^{\alpha_0}}(|k|^{\alpha_0} k)$, we have

$$\begin{aligned} |\xi| &\leq \left| \xi - |k|^{\alpha_0} k \right| + \left| |k|^{\alpha_0} k \right| \\ &< r |k|^{\alpha_0} + |k|^{\alpha_0} |k| \\ &\leq \langle k \rangle^{\alpha_0} \cdot [r + \langle k \rangle] \\ &(\text{since } \langle k \rangle \geq 1) \leq (1+r) \cdot \langle k \rangle^{\alpha_0} \langle k \rangle \\ &= (1+r) \cdot \langle k \rangle^{\frac{1}{1-\alpha}}. \end{aligned}$$

Because of $\frac{1}{1-\alpha} > 0$, we have $\langle k \rangle^{\frac{1}{1-\alpha}} \geq 1$. Together, this yields “ \lesssim ”; indeed,

$$\langle \xi \rangle \leq 1 + |\xi| \leq \langle k \rangle^{\frac{1}{1-\alpha}} + |\xi| \leq (2+r) \cdot \langle k \rangle^{\frac{1}{1-\alpha}}.$$

For the proof of “ \gtrsim ”, we distinguish two cases:

Case 1. We have $|k| \geq 2r$. This implies $|k| - r \geq |k| - \frac{|k|}{2} = \frac{|k|}{2}$ and hence

$$\begin{aligned} |\xi| &= \left| |k|^{\alpha_0} k - (|k|^{\alpha_0} k - \xi) \right| \\ &\geq \left| |k|^{\alpha_0} k \right| - \left| |k|^{\alpha_0} k - \xi \right| \\ &> |k|^{\alpha_0} |k| - r |k|^{\alpha_0} \\ &= |k|^{\alpha_0} \cdot (|k| - r) \\ &\geq \frac{1}{2} \cdot |k|^{\alpha_0} |k| = \frac{1}{2} \cdot |k|^{\frac{1}{1-\alpha}}. \end{aligned}$$

Because of $k \in \mathbb{Z}^d \setminus \{0\}$, we have $|k| \geq 1$. But this implies $\langle k \rangle \leq 1 + |k| \leq 2|k|$ and hence

$$\langle k \rangle^{\frac{1}{1-\alpha}} \leq 2^{\frac{1}{1-\alpha}} \cdot |k|^{\frac{1}{1-\alpha}} \leq 2^{1+\frac{1}{1-\alpha}} \cdot |\xi| \leq 2^{\frac{2-\alpha}{1-\alpha}} \cdot \langle \xi \rangle.$$

Case 2. We have $|k| \leq 2r$. In this case, we use the positivity of $\frac{1}{1-\alpha}$ together with $\langle \xi \rangle \geq 1$ and $\langle k \rangle \leq 1 + |k| \leq 1 + 2r$ to derive

$$\langle k \rangle^{\frac{1}{1-\alpha}} \leq (1+2r)^{\frac{1}{1-\alpha}} \leq (1+2r)^{\frac{1}{1-\alpha}} \cdot \langle \xi \rangle.$$

Together, this proves $\langle k \rangle^{\frac{1}{1-\alpha}} \leq C_0 \cdot \langle \xi \rangle$ with

$$C_0 := \max \left\{ (1+2r)^{\frac{1}{1-\alpha}}, 2^{\frac{2-\alpha}{1-\alpha}} \right\}.$$

We have thus established the first part of the lemma.

To prove $\mathcal{Q}_r^{(\alpha)}$ -moderateness of $w^{(\gamma)}$, note that $\xi \in Q_{r,k}^{(\alpha)} \cap Q_{r,\ell}^{(\alpha)}$ implies

$$\langle k \rangle \asymp \langle \xi \rangle^{1-\alpha} \asymp \langle \ell \rangle$$

and hence $1 \lesssim \langle k \rangle / \langle \ell \rangle \lesssim 1$, which yields

$$1 \lesssim (\langle k \rangle / \langle \ell \rangle)^\gamma \lesssim 1$$

and thus finally $w_k^{(\gamma)} = \langle k \rangle^\gamma \asymp \langle \ell \rangle^\gamma = w_\ell^{(\gamma)}$, where the implied constants only depend on r, α, γ . \square

Using this moderateness, we can now show that $\mathcal{Q}_r^{(\alpha)}$ is a *structured* admissible covering of \mathbb{R}^d .

Lemma 9.3. *Let $d \in \mathbb{N}$, $\alpha \in [0, 1)$ and set $\alpha_0 := \frac{\alpha}{1-\alpha}$. Let $r_1 = r_1(d, \alpha)$ as in Theorem 9.1. Then*

$$\mathcal{Q}_r^{(\alpha)} = \left(Q_{r,k}^{(\alpha)} \right)_{k \in \mathbb{Z}^d \setminus \{0\}} = \left(T_k^{(\alpha)} Q^{(r)} + b_k \right)_{k \in \mathbb{Z}^d \setminus \{0\}}$$

is a structured admissible covering of \mathbb{R}^d for each $r > r_1$, with

$$T_k^{(\alpha)} := |k|^{\alpha_0} \cdot \text{id} \quad \text{and} \quad b_k := |k|^{\alpha_0} k, \quad \text{as well as} \quad Q^{(r)} := B_r(0)$$

for each $k \in \mathbb{Z}^d \setminus \{0\}$. \blacktriangleleft

Proof. Let $s := \frac{r_1+r}{2}$, so that we have $r_1 < s < r$. Define $P := Q^{(s)} = B_s(0)$. This implies $\overline{P} := \overline{Q^{(s)}} \subset Q^{(r)} =: Q$, where P, Q are both open and bounded. Using Theorem 9.1, we see that $\mathcal{Q}_s^{(\alpha)} = (T_k^{(\alpha)}P + b_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ and $\mathcal{Q}_r^{(\alpha)} = (T_k^{(\alpha)}Q + b_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ are both admissible coverings of \mathbb{R}^d .

It remains to show that $C_{\mathcal{Q}_r^{(\alpha)}}$ is finite. To see this, let $k, \ell \in \mathbb{Z}^d \setminus \{0\}$ with $\emptyset \neq \mathcal{Q}_{r,k}^{(\alpha)} \cap \mathcal{Q}_{r,\ell}^{(\alpha)}$. Using $\langle k \rangle \leq 1 + |k| \leq 2|k|$, as well as $|\ell| \leq \langle \ell \rangle$ and $\alpha_0 \geq 0$, we see

$$\begin{aligned} \left\| (T_k^{(\alpha)})^{-1} T_\ell^{(\alpha)} \right\| &= \left(\frac{|\ell|}{|k|} \right)^{\alpha_0} \leq \left(\frac{\langle \ell \rangle}{|k|} \right)^{\alpha_0} \leq \left(2 \frac{\langle \ell \rangle}{\langle k \rangle} \right)^{\alpha_0} \\ &= 2^{\alpha_0} \cdot \frac{w_\ell^{(\alpha_0)}}{w_k^{(\alpha_0)}} \leq 2^{\alpha_0} \cdot C_{w^{(\alpha_0)}, \mathcal{Q}_r^{(\alpha)}}. \end{aligned}$$

But $C_{w^{(\alpha_0)}, \mathcal{Q}_r^{(\alpha)}}$ is finite as a consequence of Lemma 9.2. This completes the proof. \square

Lemma 9.3 (together with Theorem 3.19) implies in particular that the α -covering $\mathcal{Q}_r^{(\alpha)}$ admits a family $(\varphi_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ which is an L^p -BAPU for $\mathcal{Q}_r^{(\alpha)}$ for every $p \in (0, \infty]$. Furthermore, the weight $w^{(\gamma)}$ is $\mathcal{Q}_r^{(\alpha)}$ -moderate by Lemma 9.2. Hence, the associated decomposition spaces are well-defined.

Definition 9.4. Let $d \in \mathbb{N}$ and $0 \leq \alpha < 1$ be arbitrary. Choose $r_1 = r_1(d, \alpha)$ as in Theorem 9.1. For each $r > r_1$, and $p, q \in (0, \infty]$, as well as $\gamma \in \mathbb{R}$, we define the **Fourier-side α -modulation space** with integrability exponents p, q and weight-exponent γ as

$$\mathcal{FM}_{\gamma, \alpha}^{p, q}(\mathbb{R}^d) := \mathcal{D}_{\mathcal{F}} \left(\mathcal{Q}_r^{(\alpha)}, L^p, \ell_{w^{(\gamma/(1-\alpha))}}^q \right),$$

where the weight $w^{(\gamma/(1-\alpha))} = \left(\langle k \rangle^{\gamma/(1-\alpha)} \right)_{k \in \mathbb{Z}^d \setminus \{0\}}$ is defined as in Lemma 9.2. \blacktriangleleft

Remark. Observe that the parameter r is suppressed on the left-hand side of the definition above. But since any two coverings $\mathcal{Q}_r^{(\alpha)}, \mathcal{Q}_s^{(\alpha)}$ (with $r, s > r_1(d, \alpha)$) use the *same* family of normalizations, it follows that every L^p -BAPU Φ for $\mathcal{Q}_r^{(\alpha)}$ is also an L^p -BAPU for $\mathcal{Q}_s^{(\alpha)}$, at least for $s \geq r$, which we can always assume by symmetry. With this choice of the BAPUs for both spaces, we get

$$\|\cdot\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}_r^{(\alpha)}, L^p, \ell_{w^{(\gamma/(1-\alpha))}}^q)} = \|\cdot\|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{Q}_s^{(\alpha)}, L^p, \ell_{w^{(\gamma/(1-\alpha))}}^q)},$$

which implies that both spaces coincide. Since different choices of the BAPU yield the same space with equivalent quasi-norms (cf. Corollary 3.18), we see that the right-hand side of the above definition is independent of the choice of $r > r_1(d, \alpha)$, with equivalent quasi-norms for different choices.

We will also show below (cf. Corollary 9.16) that the Fourier transform restricts to an isomorphism

$$\mathcal{F} : M_{\gamma, \alpha}^{p, q}(\mathbb{R}^d) \rightarrow \mathcal{FM}_{\gamma, \alpha}^{p, q}(\mathbb{R}^d),$$

where the (space-side) α -modulation space on the left-hand side is defined as in [2, Definition 2.4]. This justifies the notation $\mathcal{FM}_{\gamma, \alpha}^{p, q}(\mathbb{R}^d)$. \blacklozenge

As the next step in our program, we will estimate the cardinalities of the “intersection sets” J_i , where $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ are both certain α -coverings, for different values of α . As a byproduct, we will then see (using Lemma 2.12) that $\mathcal{Q}_r^{(\alpha)}$ is almost subordinate to $\mathcal{Q}_s^{(\beta)}$ for $\alpha \leq \beta$.

Lemma 9.5. Let $d \in \mathbb{N}$ and $\alpha, \beta \in [0, 1)$ with $\alpha \leq \beta$. Let $r_1 = r_1(d, \alpha)$ and $r_2 = r_1(d, \beta)$ as in Theorem 9.1 and let $r > r_1$ and $s > r_2$.

There is a constant $C = C(d, \alpha, \beta, r, s) > 0$ with

$$1 \leq \left| \left\{ \ell \in \mathbb{Z}^d \setminus \{0\} \mid \mathcal{Q}_{s, \ell}^{(\beta)} \cap \mathcal{Q}_{r, k}^{(\alpha)} \neq \emptyset \right\} \right| \leq C \quad (9.1)$$

for all $k \in \mathbb{Z}^d \setminus \{0\}$.

In particular, $\mathcal{Q}_r^{(\alpha)}$ is almost subordinate to $\mathcal{Q}_s^{(\beta)}$. \blacktriangleleft

Proof. To be consistent with our usual notation, set $\mathcal{Q} := \mathcal{Q}_r^{(\alpha)}$ and $\mathcal{P} := \mathcal{Q}_s^{(\beta)}$. Then, the intersection set J_k is given by

$$J_k = \left\{ \ell \in \mathbb{Z}^d \setminus \{0\} \mid \mathcal{Q}_{s, \ell}^{(\beta)} \cap \mathcal{Q}_{r, k}^{(\alpha)} \neq \emptyset \right\}$$

for all $k \in \mathbb{Z}^d \setminus \{0\}$. Once we have established estimate (9.1), we have shown that \mathcal{Q} is *weakly* subordinate to \mathcal{P} , cf. Definition 2.10. But since all sets in \mathcal{Q} and \mathcal{P} are open balls, which are open and path-connected, Corollary 2.13 easily implies that $\mathcal{Q} = \mathcal{Q}_r^{(\alpha)}$ is almost subordinate to $\mathcal{P} = \mathcal{Q}_s^{(\beta)}$, since \mathcal{Q}, \mathcal{P} are both coverings of all of \mathbb{R}^d .

It thus remains to establish equation (9.1). To this end, set $\alpha_0 := \frac{\alpha}{1-\alpha}$ and $\beta_0 := \frac{\beta}{1-\beta}$. For $m \in \mathbb{Z}^d \setminus \{0\}$ and $\gamma \in [0, 1)$, let us denote the γ -**normalized version** of m by $m^{(\gamma)} := |m|^{\gamma_0} m$ with $\gamma_0 := \frac{\gamma}{1-\gamma}$. By definition, we have $Q_{t,m}^{(\gamma)} = B_{|m|^{\gamma_0}}(m^{(\gamma)})$ for all $t > 0$ and $m \in \mathbb{Z}^d \setminus \{0\}$.

Lemma 9.2 implies

$$\begin{aligned} \langle k \rangle^{\frac{1}{1-\alpha}} &\asymp \langle \xi \rangle & \forall k \in \mathbb{Z}^d \setminus \{0\} \text{ and } \xi \in Q_{r,k}^{(\alpha)}, \\ \langle \ell \rangle^{\frac{1}{1-\beta}} &\asymp \langle \xi \rangle & \forall \ell \in \mathbb{Z}^d \setminus \{0\} \text{ and } \xi \in Q_{s,\ell}^{(\beta)}, \end{aligned} \quad (9.2)$$

where the implied constants only depend on d, α, β, r, s .

The lower estimate in equation (9.1) is trivial, because of

$$\emptyset \neq Q_{r,k}^{(\alpha)} \subset \mathbb{R}^d = \bigcup_{\ell \in \mathbb{Z}^d \setminus \{0\}} Q_{s,\ell}^{(\beta)}.$$

To prove the upper estimate, fix $k \in \mathbb{Z}^d \setminus \{0\}$, some $\ell_0 \in J_k$ and some $\xi_0 \in Q_{s,\ell_0}^{(\beta)} \cap Q_{r,k}^{(\alpha)}$. Using equation (9.2), this yields

$$|k|^{\alpha_0} \leq \left(\langle k \rangle^{\frac{1}{1-\alpha}} \right)^\alpha \lesssim \langle \xi_0 \rangle^\alpha \lesssim \left(\langle \ell_0 \rangle^{\frac{1}{1-\beta}} \right)^\alpha \stackrel{(\dagger)}{\leq} \langle \ell_0 \rangle^{\frac{\beta}{1-\beta}} = \langle \ell_0 \rangle^{\beta_0}$$

where the step marked with (\dagger) is justified by $\langle \ell_0 \rangle \geq 1$ and $\alpha \leq \beta$. As usual, the implied constants only depend on d, α, β, r, s .

Note that for each $\ell \in J_k$, there is some $\xi \in Q_{r,k}^{(\alpha)} \cap Q_{s,\ell}^{(\beta)}$. Hence, $\xi, \xi_0 \in Q_{r,k}^{(\alpha)}$ which yields $\langle \xi \rangle \asymp \langle k \rangle^{1/(1-\alpha)} \asymp \langle \xi_0 \rangle$ by equation (9.2). Now, an application of equation (9.2), together with $\langle \ell \rangle \leq 1 + |\ell| \leq 2|\ell|$, leads to

$$\langle \ell_0 \rangle^{\beta_0} = \langle \ell_0 \rangle^{\frac{\beta}{1-\beta}} \lesssim \langle \xi_0 \rangle^\beta \lesssim \langle \xi \rangle^\beta \lesssim \left(\langle \ell \rangle^{\frac{1}{1-\beta}} \right)^\beta \lesssim |\ell|^{\beta_0} \quad (9.3)$$

and

$$|\ell|^{\beta_0} \leq \left(\langle \ell \rangle^{\frac{1}{1-\beta}} \right)^\beta \lesssim \langle \xi \rangle^\beta \lesssim \langle \xi_0 \rangle^\beta \lesssim \left(\langle \ell_0 \rangle^{\frac{1}{1-\beta}} \right)^\beta = \langle \ell_0 \rangle^{\beta_0}.$$

For arbitrary $\eta \in Q_{s,\ell}^{(\beta)}$, a combination of the estimates from the previous two paragraphs shows

$$\begin{aligned} \left| \eta - |\ell_0|^{\beta_0} \ell_0 \right| &\leq \left| \eta - \ell^{(\beta_0)} \right| + \left| \ell^{(\beta_0)} - \xi \right| + \left| \xi - k^{(\alpha_0)} \right| + \left| k^{(\alpha_0)} - \xi_0 \right| + \left| \xi_0 - \ell_0^{(\beta_0)} \right| \\ &< s \cdot |\ell|^{\beta_0} + s \cdot |\ell|^{\beta_0} + r \cdot |k|^{\alpha_0} + r \cdot |k|^{\alpha_0} + s \cdot |\ell_0|^{\beta_0} \\ &\lesssim (s+r) \cdot \langle \ell_0 \rangle^{\beta_0} \end{aligned}$$

and thus $Q_{s,\ell}^{(\beta)} \subset B_{C_1 \langle \ell_0 \rangle^{\beta_0}}(|\ell_0|^{\beta_0} \ell_0)$ for all $\ell \in J_k$ and some constant $C_1 = C_1(d, \alpha, \beta, r, s)$.

But admissibility of $\mathcal{Q}_s^{(\beta)}$ yields a constant $C_2 = C_2(d, \beta, s) > 0$ with

$$\sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \mathbb{1}_{Q_{s,\ell}^{(\beta)}} \leq C_2.$$

In fact, we can take $C_2 = N_{\mathcal{Q}_s^{(\beta)}}$, cf. the remark after Definition 2.3. Together with the above inclusion, we conclude

$$\sum_{\ell \in J_k} \mathbb{1}_{Q_{s,\ell}^{(\beta)}} \leq C_2 \cdot \mathbb{1}_{B_{C_1 \langle \ell_0 \rangle^{\beta_0}}(|\ell_0|^{\beta_0} \ell_0)}. \quad (9.4)$$

Furthermore, equation (9.3) yields

$$\lambda \left(Q_{s,\ell}^{(\beta)} \right) = \lambda \left(B_1(0) \right) \cdot s^d \cdot |\ell|^{d\beta_0} \gtrsim s^d \cdot \langle \ell_0 \rangle^{d\beta_0}$$

for all $\ell \in J_k$. Thus, integration of estimate (9.4) leads to

$$\begin{aligned} s^d \cdot |J_k| \cdot \langle \ell_0 \rangle^{d\beta_0} &\lesssim \sum_{\ell \in J_k} \lambda \left(O_{s,\ell}^{(\beta)} \right) \\ &= \int \sum_{\ell \in J_k} \mathbb{1}_{Q_{s,\ell}^{(\beta)}}(\xi) \, d\xi \\ (\text{eq. (9.4)}) &\leq C_2 \cdot \lambda \left(B_{C_1 \langle \ell_0 \rangle^{\beta_0}}(|\ell_0|^{\beta_0} \ell_0) \right) \\ &\lesssim C_2 C_1^d \cdot \langle \ell_0 \rangle^{d\beta_0}, \end{aligned}$$

This establishes the upper estimate in equation (9.1) and thereby completes the proof. \square

As we just saw, the α -covering $\mathcal{Q}_r^{(\alpha)}$ is almost subordinate to the β -covering $\mathcal{Q}_s^{(\beta)}$, for $\alpha \leq \beta$. Even more is true:

Lemma 9.6. *Let $d \in \mathbb{N}$ and $\alpha, \beta \in [0, 1)$ with $\alpha \leq \beta$ be arbitrary. Choose $r_1 = r_1(d, \alpha)$ and $r_2 = r_1(d, \beta)$ as in Theorem 9.1 and let $r > r_1$ as well as $s > r_2$.*

Then $\mathcal{Q}_r^{(\alpha)}$ is almost subordinate to $\mathcal{Q}_s^{(\beta)}$ and relatively $\mathcal{Q}_s^{(\beta)}$ -moderate. More precisely,

$$C^{-1} \cdot \langle \ell \rangle^{d \frac{\alpha}{1-\beta}} \leq |\det(T_k^{(\alpha)})| \leq C \cdot \langle \ell \rangle^{d \frac{\alpha}{1-\beta}} \quad (9.5)$$

holds for all $k, \ell \in \mathbb{Z}^d \setminus \{0\}$ with $Q_{r,k}^{(\alpha)} \cap Q_{s,\ell}^{(\beta)} \neq \emptyset$ and some constant $C = C(d, \alpha, \beta, r, s) \geq 1$.

Finally, let $\gamma \in \mathbb{R}$ be arbitrary. Then, there is $L = L(d, \alpha, \beta, \gamma, r, s) \geq 1$ with

$$L^{-1} \cdot w_\ell^{(\gamma \cdot \frac{1-\alpha}{1-\beta})} \leq w_k^{(\gamma)} \leq L \cdot w_\ell^{(\gamma \cdot \frac{1-\alpha}{1-\beta})} \quad (9.6)$$

for all $k, \ell \in \mathbb{Z}^d \setminus \{0\}$ with $Q_{r,k}^{(\alpha)} \cap Q_{s,\ell}^{(\beta)} \neq \emptyset$. Here, $w^{(\gamma)}$ is defined as in Lemma 9.2. \blacktriangleleft

Proof. Lemma 9.5 shows that $\mathcal{Q}_r^{(\alpha)}$ is almost subordinate to $\mathcal{Q}_s^{(\beta)}$, so that we only have to establish estimates (9.5) and (9.6).

To this end, let $k, \ell \in \mathbb{Z}^d \setminus \{0\}$ with $\emptyset \neq Q_{r,k}^{(\alpha)} \cap Q_{s,\ell}^{(\beta)} \ni \xi$. Lemma 9.2 allows us to conclude

$$\langle k \rangle^{\frac{1}{1-\alpha}} \asymp \langle \xi \rangle \asymp \langle \ell \rangle^{\frac{1}{1-\beta}},$$

where the implied constants only depend on d, α, β, r, s . If implied constants also depend on γ , this is indicated by \asymp_γ and \lesssim_γ . The above implies $\langle k \rangle^{\frac{1}{1-\alpha}} / \langle \ell \rangle^{\frac{1}{1-\beta}} \asymp 1$ and hence

$$1 \lesssim_\gamma \left(\frac{\langle k \rangle^{\frac{1}{1-\alpha}}}{\langle \ell \rangle^{\frac{1}{1-\beta}}} \right)^{(1-\alpha)\gamma} \lesssim_\gamma 1,$$

which means $w_k^{(\gamma)} = \langle k \rangle^\gamma \asymp_\gamma \langle \ell \rangle^{\gamma \cdot \frac{1-\alpha}{1-\beta}} = w_\ell^{(\gamma \cdot \frac{1-\alpha}{1-\beta})}$. Hence, equation (9.6) is established.

Finally, the considerations from above (with $\gamma = d \cdot \alpha_0$) also show that $\mathcal{Q}_r^{(\alpha)} = (T_k^{(\alpha)} Q^{(r)} + b_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ is relatively moderate with respect to $\mathcal{Q}_s^{(\beta)}$, because of

$$\begin{aligned} |\det(T_k^{(\alpha)})| &= |\det(|k|^{\alpha_0} \text{id})| = |k|^{d \cdot \alpha_0} \\ &\asymp \langle k \rangle^{d \cdot \alpha_0} = w_k^{(d \cdot \alpha_0)} \asymp \langle \ell \rangle^{d \cdot \alpha_0 \cdot \frac{1-\alpha}{1-\beta}} = \langle \ell \rangle^{d \cdot \frac{\alpha}{1-\beta}} \end{aligned}$$

for all $k \in \mathbb{Z}^d \setminus \{0\}$ with $Q_{r,k}^{(\alpha)} \cap Q_{s,\ell}^{(\beta)} \neq \emptyset$. This also establishes equation (9.5). \square

After the above analysis of the geometric relationship between the different α -coverings, it is now essentially straightforward to derive the announced embeddings between α -modulation spaces with different values of α .

The “limit case” $\alpha = 1$ —which corresponds to (inhomogeneous) Besov spaces—will be treated below, cf. Theorem 9.13. Finally, we remark that the present theorem is essentially identical to [23, Theorem 6.1.7] from my PhD thesis.

Theorem 9.7. Let $d \in \mathbb{N}$ and $\alpha, \beta \in [0, 1)$ with $\alpha \leq \beta$. Finally, let $p_1, p_2, q_1, q_2 \in (0, \infty]$, as well as $\gamma_1, \gamma_2 \in \mathbb{R}$. Define

$$\begin{aligned}\gamma^{(0)} &:= \alpha \left(\frac{1}{p_2} - \frac{1}{p_1} \right) + (\alpha - \beta) \left(\frac{1}{p_2^\nabla} - \frac{1}{q_1} \right)_+, \\ \gamma^{(1)} &:= \alpha \left(\frac{1}{p_2} - \frac{1}{p_1} \right) + (\alpha - \beta) \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\Delta}} \right)_+.\end{aligned}$$

The map

$$\iota : \mathcal{FM}_{\gamma_1, \alpha}^{p_1, q_1}(\mathbb{R}^d) \rightarrow \mathcal{FM}_{\gamma_2, \beta}^{p_2, q_2}(\mathbb{R}^d), f \mapsto f$$

is well-defined and bounded if and only if we have $p_1 \leq p_2$, as well as

$$\begin{cases} \gamma_2 \leq \gamma_1 + d \cdot \gamma^{(0)}, & \text{if } q_1 \leq q_2, \\ \gamma_2 < \gamma_1 + d \cdot \left(\gamma^{(0)} + (1 - \beta) \left(\frac{1}{q_1} - \frac{1}{q_2} \right) \right), & \text{if } q_1 > q_2. \end{cases}$$

Conversely, the map

$$\theta : \mathcal{FM}_{\gamma_1, \beta}^{p_1, q_1}(\mathbb{R}^d) \rightarrow \mathcal{FM}_{\gamma_2, \alpha}^{p_2, q_2}(\mathbb{R}^d), f \mapsto f$$

is well-defined and bounded if and only if we have $p_1 \leq p_2$, as well as

$$\begin{cases} \gamma_2 \leq \gamma_1 + d \cdot \gamma^{(1)}, & \text{if } q_1 \leq q_2, \\ \gamma_2 < \gamma_1 + d \cdot \left(\gamma^{(1)} + (1 - \beta) \left(\frac{1}{q_1} - \frac{1}{q_2} \right) \right), & \text{if } q_1 > q_2. \end{cases} \quad \blacktriangleleft$$

Remark 9.8. In case of $q_1 = q_2 = q$ and $p_1 = p_2 = p$, the first condition reduces to

$$\gamma_2 + d(\beta - \alpha) \left(\frac{1}{p^\nabla} - \frac{1}{q} \right)_+ \leq \gamma_1,$$

which is identical to the condition given by Han and Wang in [13, Theorem 4.1].

Analogously, the second condition reduces to

$$\gamma_2 + d(\beta - \alpha) \left(\frac{1}{q} - \frac{1}{p^{\pm\Delta}} \right)_+ \leq \gamma_1,$$

which also coincides with the condition stated in [13, Theorem 4.1]. The verification that the stated conditions are equivalent to the ones given by Han and Wang is straightforward, but slightly tedious, since Han and Wang use a different notation. \blacklozenge

Proof. Let $r_1 = r_1(d, \alpha)$ and $r_2 = r_1(d, \beta)$ as in Theorem 9.1, choose $r > r_1$, as well as $s > r_2$ and recall the definition of the spaces $\mathcal{FM}_{\gamma, \alpha}^{p, q}(\mathbb{R}^d) = \mathcal{D}_{\mathcal{F}} \left(\mathcal{Q}_r^{(\alpha)}, L^p, \ell_{w^{(\gamma/(1-\alpha))}}^q \right)$ from Definition 9.4.

To analyze boundedness of ι , we define $\mathcal{Q} := \mathcal{Q}_r^{(\alpha)}$ and $\mathcal{P} := \mathcal{Q}_s^{(\beta)}$, as well as $w := w^{(\gamma_1/(1-\alpha))}$ and $v := w^{(\gamma_2/(1-\beta))}$, so that we have $\mathcal{FM}_{\gamma_1, \alpha}^{p_1, q_1}(\mathbb{R}^d) = \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$ and $\mathcal{FM}_{\gamma_2, \beta}^{p_2, q_2}(\mathbb{R}^d) = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$.

Lemma 9.3 shows that \mathcal{Q}, \mathcal{P} are structured admissible coverings of \mathbb{R}^d , which thus satisfy the standing assumptions from Section 7 (i.e., Assumption 7.1). Furthermore, w is \mathcal{Q} -moderate and v is \mathcal{P} -moderate, thanks to Lemma 9.2, so that Assumption 7.1 is completely satisfied.

Finally, since we have $\alpha \leq \beta$, Lemma 9.6 shows that \mathcal{Q} is almost subordinate to \mathcal{P} and that \mathcal{Q} and w are relatively \mathcal{P} -moderate. Thus, all assumptions of part (4) of Theorem 7.4 are satisfied. Note that the embedding ι from Theorem 7.4 satisfies $\iota f = f$ for all $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$, since \mathcal{Q}, \mathcal{P} both cover the same set $\mathcal{O} = \mathbb{R}^d = \mathcal{O}'$. Thus, all in all, we conclude that ι is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if

$$K := \left\| \left(\frac{v_j}{w_{i_j}} \cdot |\det T_{i_j}|^{\frac{1}{p_1} - \frac{1}{p_2} - s} \cdot |\det S_j|^s \right)_{j \in J} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}$$

is finite, where $s = \left(\frac{1}{p_2^\nabla} - \frac{1}{q_1} \right)_+$ and where for each $j \in J = \mathbb{Z}^d \setminus \{0\}$, some $i_j \in I_j$, i.e. with $Q_{r, i_j}^{(\alpha)} \cap Q_{s, j}^{(\beta)} \neq \emptyset$ is selected. According to Lemma 9.6, this implies

$$|\det T_{i_j}| \asymp \langle j \rangle^{d \frac{\alpha}{1-\beta}} \quad \text{and} \quad w_{i_j} = w_{i_j}^{(\gamma_1/(1-\alpha))} \asymp w_j^{(\frac{\gamma_1}{1-\alpha}, \frac{1-\alpha}{1-\beta})} = w_j^{(\gamma_1/(1-\beta))} = \langle j \rangle^{\frac{\gamma_1}{1-\beta}}.$$

Since we also have $|\det S_j| = |j|^{d\frac{\beta}{1-\beta}}$ and $v_j = w_j^{(\gamma_2/(1-\beta))} = \langle j \rangle^{\frac{\gamma_2}{1-\beta}}$, we get

$$\begin{aligned} K &\asymp \left\| \left(\langle j \rangle^{\frac{\gamma_2-\gamma_1}{1-\beta}} \cdot \langle j \rangle^{d\frac{\alpha}{1-\beta}(\frac{1}{p_1}-\frac{1}{p_2}-s)} \cdot |j|^{ds\frac{\beta}{1-\beta}} \right)_{j \in \mathbb{Z}^d \setminus \{0\}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ &\asymp \left\| \left(\langle j \rangle^{\frac{1}{1-\beta}[(\gamma_2-\gamma_1)+d\alpha(p_1^{-1}-p_2^{-1}-s)+d\beta s]} \right)_{j \in \mathbb{Z}^d \setminus \{0\}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}. \end{aligned}$$

Recall from equation (4.3) that $q_2 \cdot (q_1/q_2)'$ is finite if and only if $q_2 < q_1$. Hence, there are two cases:

Case 1: $q_1 \leq q_2$. Here, we have $q_2 \cdot (q_1/q_2)' = \infty$, so that we get the following equivalence:

$$\begin{aligned} K < \infty &\iff \frac{1}{1-\beta} [(\gamma_2 - \gamma_1) + d\alpha(p_1^{-1} - p_2^{-1} - s) + d\beta s] \leq 0 \\ &\iff \gamma_2 + d[s(\beta - \alpha) + \alpha(p_1^{-1} - p_2^{-1})] \leq \gamma_1 \\ &\iff \gamma_2 \leq \gamma_1 + d[s(\alpha - \beta) + \alpha(p_2^{-1} - p_1^{-1})] \\ &\iff \gamma_2 \leq \gamma_1 + d \cdot \gamma^{(0)}. \end{aligned}$$

This completes the proof for $q_1 \leq q_2$.

Case 2: $q_1 > q_2$. Here, we have $t := q_2 \cdot (q_1/q_2)' < \infty$. But for arbitrary $\varrho \in \mathbb{R}$ and $t \in (0, \infty)$, it is an elementary fact that

$$\left\| (\langle j \rangle^{\varrho})_{j \in \mathbb{Z}^d \setminus \{0\}} \right\|_{\ell^t} < \infty \iff t \cdot \varrho < -d \iff \varrho < -\frac{d}{t}.$$

In our case, equation (4.2) yields $\frac{1}{t} = \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+ = \frac{1}{q_2} - \frac{1}{q_1}$ and we have

$$\begin{aligned} \varrho &= \frac{1}{1-\beta} [(\gamma_2 - \gamma_1) + d\alpha(p_1^{-1} - p_2^{-1} - s) + d\beta s] \\ &= \frac{1}{1-\beta} [(\gamma_2 - \gamma_1) + d(s(\beta - \alpha) + \alpha(p_1^{-1} - p_2^{-1}))], \end{aligned}$$

so that we get all in all that $K < \infty$ is equivalent to

$$\begin{aligned} &\frac{1}{1-\beta} [(\gamma_2 - \gamma_1) + d(s(\beta - \alpha) + \alpha(p_1^{-1} - p_2^{-1}))] \stackrel{!}{<} -d \left(\frac{1}{q_2} - \frac{1}{q_1} \right) \\ &\iff (\gamma_2 - \gamma_1) + d(s(\beta - \alpha) + \alpha(p_1^{-1} - p_2^{-1})) \stackrel{!}{<} d(1-\beta) \left(\frac{1}{q_1} - \frac{1}{q_2} \right) \\ &\iff \gamma_2 \stackrel{!}{<} \gamma_1 + d(s(\alpha - \beta) + \alpha(p_2^{-1} - p_1^{-1})) + d(1-\beta) \left(\frac{1}{q_1} - \frac{1}{q_2} \right) \\ &\iff \gamma_2 \stackrel{!}{<} \gamma_1 + d \left(\gamma^{(0)} + (1-\beta) \left(\frac{1}{q_1} - \frac{1}{q_2} \right) \right). \end{aligned}$$

This completes the proof for $q_1 > q_2$.

Now, we consider boundedness of θ . To this end, we define $\mathcal{Q} := \mathcal{Q}_s^{(\beta)}$ and $\mathcal{P} := \mathcal{Q}_r^{(\alpha)}$, as well as $w := w^{(\gamma_1/(1-\beta))}$ and $v := w^{(\gamma_2/(1-\alpha))}$, so that we have $\mathcal{FM}_{\gamma_1, \beta}^{p_1, q_1}(\mathbb{R}^d) = \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$ and $\mathcal{FM}_{\gamma_2, \alpha}^{p_2, q_2}(\mathbb{R}^d) = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$.

Exactly as for boundedness of ι , we see that Assumption 7.1 is fulfilled. Furthermore, since $\alpha \leq \beta$, Lemma 9.6 shows that $\mathcal{P} = \mathcal{Q}_r^{(\alpha)}$ is almost subordinate to $\mathcal{Q} = \mathcal{Q}_s^{(\beta)}$. The same lemma also shows that \mathcal{P} and v are relatively \mathcal{Q} -moderate, so that all assumptions of part (4) of Theorem 7.2 are satisfied.

As in the case of ι , this shows that θ is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if

$$K := \left\| \left(\frac{v_{j_i}}{w_i} \cdot |\det T_i|^s \cdot |\det S_{j_i}|^{\frac{1}{p_1} - \frac{1}{p_2} - s} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} < \infty$$

holds, where $s = \left(\frac{1}{q_2} - \frac{1}{p_1}\right)_+$ and where for each index $i \in I = \mathbb{Z}^d \setminus \{0\}$, some $j_i \in J_i$, i.e. with $\mathcal{Q}_{r, j_i}^{(\alpha)} \cap \mathcal{Q}_{s, i}^{(\beta)} \neq \emptyset$ is selected.

In view of Lemma 9.6, this implies

$$|\det S_{j_i}| \asymp \langle i \rangle^{d \frac{\alpha}{1-\beta}} \quad \text{and} \quad v_{j_i} = w_{j_i}^{(\gamma_2/(1-\alpha))} \asymp w_i^{(\frac{\gamma_2}{1-\alpha} \cdot \frac{1-\alpha}{1-\beta})} = w_i^{(\gamma_2/(1-\beta))} = \langle i \rangle^{\frac{\gamma_2}{1-\beta}},$$

so that we get

$$\begin{aligned} K &\asymp \left\| \left(\langle i \rangle^{\frac{\gamma_2-\gamma_1}{1-\beta}} \cdot \langle i \rangle^{d \frac{\alpha}{1-\beta} (p_1^{-1} - p_2^{-1} - s)} \cdot |i|^{ds \frac{\beta}{1-\beta}} \right)_{i \in \mathbb{Z}^d \setminus \{0\}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ &\asymp \left\| \left(\langle i \rangle^{\frac{1}{1-\beta} [(\gamma_2-\gamma_1) + d(s(\beta-\alpha) + \alpha(p_1^{-1} - p_2^{-1}))]} \right)_{i \in \mathbb{Z}^d \setminus \{0\}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}. \end{aligned}$$

The remainder of the proof is entirely analogous to the considerations for ι and is hence omitted. \square

In summary, we have shown that the embedding results for α -modulation spaces obtained in [13, Theorem 4.1] could be derived with ease using our more general approach. We remark that our approach can also handle the case $(p_1, q_1) \neq (p_2, q_2)$ which is not considered in [13, Theorem 4.1]. Furthermore, the most tedious parts of our derivation, namely the proof that $\mathcal{Q}_r^{(\alpha)}$ is almost subordinate to $\mathcal{Q}_s^{(\beta)}$ for $\alpha \leq \beta$ and the proof of $\mathcal{Q}_s^{(\beta)}$ -relative moderateness of $\mathcal{Q}_r^{(\alpha)}$ and $w^{(\gamma)}$, are used in [13] without much justification, see equations (4.4) and (4.6) of that paper.

In the next subsection, we will begin with our study of Besov spaces. Precisely, we will extend Theorem 9.7 to include the case $\beta = 1$, which shows that the results for embeddings between α -modulation spaces and inhomogeneous Besov spaces obtained in [13, Theorem 4.2] are also special cases of our approach—in fact our approach extends these results to the case $(p_1, q_1) \neq (p_2, q_2)$.

9.2. Embeddings between α -modulation spaces and Besov spaces. It is common to consider the (inhomogeneous) Besov spaces $B_{\gamma,q}^{p,q}(\mathbb{R}^d)$ as the “limit case” of α -modulation spaces for $\alpha \uparrow 1$, i.e. to set $M_{\gamma,1}^{p,q}(\mathbb{R}^d) := B_{\gamma,q}^{p,q}(\mathbb{R}^d)$. In this subsection, we want to extend Theorem 9.7 to include the case $\beta = 1$. To this end, we first have to define the “Fourier-side” α -modulation spaces for $\alpha = 1$, i.e., the “Fourier-side” (inhomogeneous) Besov spaces.

As a further result, we will in this subsection that the Fourier transform restricts an isomorphism of Quasi-Banach spaces

$$\mathcal{F} : M_{\gamma,\alpha}^{p,q}(\mathbb{R}^d) \rightarrow \mathcal{D}_{\mathcal{F}} \left(\mathcal{Q}_r^{(\alpha)}, L^p, \ell_{w^{(\gamma/(1-\alpha))}}^q \right) = \mathcal{F} M_{\gamma,\alpha}^{p,q}(\mathbb{R}^d),$$

thereby justifying our nomenclature. As a consequence, the characterization of the embeddings for the Fourier-side α -modulation spaces from Theorem 9.7 readily extends to the “usual” (space-side) α -modulation spaces.

But let us begin by defining the inhomogeneous Besov covering:

Definition 9.9. For $d \in \mathbb{N}$ and $n \in \mathbb{N}$, define

$$B_n := B_{2^{n+2}}(0) \setminus \overline{B_{2^{n-2}}}(0) \subset \mathbb{R}^d$$

and set $B_0 := B_4(0) \subset \mathbb{R}^d$. The family $\mathcal{B} := (B_n)_{n \in \mathbb{N}_0}$ is called the **inhomogeneous Besov covering** of \mathbb{R}^d . \blacktriangleleft

The following lemma (partly) justifies our nomenclature, by showing that \mathcal{B} is indeed an (almost structured) covering of \mathbb{R}^d .

Lemma 9.10. *The inhomogeneous Besov covering $\mathcal{B} = (T_n B'_n + b_n)_{n \in \mathbb{N}_0}$ is an almost structured covering of \mathbb{R}^d , with standardizations given by*

$$T_n := 2^n \cdot \text{id}, \quad b_n := 0 \quad \text{and} \quad B'_n := \begin{cases} B_4(0) \setminus \overline{B_{1/4}}(0), & \text{if } n \in \mathbb{N}, \\ B_4(0), & \text{if } n = 0. \end{cases}$$

Furthermore, for any $\gamma \in \mathbb{R}$, the weight $v^{(\gamma)} := (2^{\gamma n})_{n \in \mathbb{N}_0}$ is \mathcal{B} -moderate. \blacktriangleleft

Proof. The given T_n, B'_n and b_n indeed satisfy $B_n = T_n B'_n + b_n$ for all $n \in \mathbb{N}_0$. Furthermore, if we set

$$P_n := \begin{cases} B_2(0) \setminus \overline{B_{1/2}}(0), & \text{if } n \in \mathbb{N}, \\ B_2(0), & \text{if } n = 0, \end{cases}$$

then all P_n and all B'_n are open and bounded, the sets $\{P_n \mid n \in \mathbb{N}_0\}$ and $\{B'_n \mid n \in \mathbb{N}_0\}$ are finite and we have $\overline{P_n} \subset B'_n$ for all $n \in \mathbb{N}_0$.

Furthermore, $\mathbb{R}^d = \bigcup_{n \in \mathbb{N}_0} (T_n P_n + b_n)$. Indeed, for $\xi \in \mathbb{R}^d$, choose $n \in \mathbb{N}_0$ maximal with $2^n \leq |\xi|$. This implies $2^n \leq |\xi| < 2^{n+1}$. In case of $n = 0$, this yields $|\xi| < 2$ and hence $\xi \in B_2(0) = T_0 P_0 + b_0$. Otherwise, for $n \in \mathbb{N}$, we have $2^{n-1} < 2^n \leq |\xi| < 2^{n+1}$, i.e. $\xi \in 2^n \cdot (B_2(0) \setminus \overline{B_{1/2}}(0)) = T_n P_n + b_n$, as desired.

Thus, recalling the definition of an almost structured covering (Definition 2.5), it remains to show that \mathcal{B} is admissible and that $C_{\mathcal{B}}$ is finite. To this end, fix $n \in \mathbb{N}$ and assume that there is some $m \in \mathbb{N}$ with $\emptyset \neq B_n \cap B_m$. If we choose $\xi \in B_n \cap B_m$, we thus get

$$2^{m-2} < |\xi| < 2^{n+2} \quad \text{and} \quad 2^{n-2} < |\xi| < 2^{m+2}.$$

In particular, $m-2 < n+2$ and $n-2 < m+2$, i.e. $n-4 < m < n+4$. Since all these quantities are integers, we get $n-3 \leq m \leq n+3$. All in all, we derive

$$n^* \subset \{0\} \cup \{n-3, \dots, n+3\} \quad \forall n \in \mathbb{N} \quad (9.7)$$

and hence $|n^*| \leq 8$. Finally, for $m \in \mathbb{N}$ with $\xi \in B_0 \cap B_m \neq \emptyset$, we get $2^{m-2} < |\xi| < 4 = 2^2$ and hence $m-2 < 2$, i.e. $m \leq 3$. We have thus shown $0^* \subset \{0, 1, 2, 3\}$ and hence $|0^*| \leq 4$. All in all, we get $N_{\mathcal{B}} = \sup_{n \in \mathbb{N}_0} |n^*| \leq 8 < \infty$, so that \mathcal{B} is an admissible covering of \mathbb{R}^d .

It remains to show that

$$C_{\mathcal{B}} := \sup_{n \in \mathbb{N}_0} \sup_{m \in n^*} \|T_n^{-1} T_m\| = \sup_{n \in \mathbb{N}_0} \sup_{m \in n^*} 2^{m-n}$$

is finite. This is indeed the case: For $n \in \mathbb{N}$, we saw in equation (9.7) that $n^* \subset \{0, \dots, n+3\}$, so that every $m \in n^*$ satisfies $2^{m-n} \leq 2^3 = 8$. Finally, for $n = 0$ and $m \in n^* = 0^* \subset \{0, 1, 2, 3\}$, we also have $2^{m-n} \leq 2^3 = 8$, so that $C_{\mathcal{B}} \leq 8 < \infty$. All in all, we have thus shown that \mathcal{B} is an almost structured covering of \mathbb{R}^d .

As an almost structured covering, \mathcal{B} is in particular a semi-structured covering, so that equation (3.11) shows that the weight $(|\det T_n|)_{n \in \mathbb{N}_0} = (2^{nd})_{n \in \mathbb{N}_0}$ is \mathcal{B} -moderate. But if $w = (w_n)_{n \in \mathbb{N}_0}$ is \mathcal{B} -moderate, it is not hard to see that the same holds for $(w_n^{\varrho})_{n \in \mathbb{N}_0}$, for arbitrary $\varrho \in \mathbb{R}$. If we apply this with $\varrho = \frac{\gamma}{d}$, we see that $v^{(\gamma)}$ is \mathcal{B} -moderate, as claimed. \square

Recall from Theorem 3.19 that every almost structured covering admits a subordinate partition of unity which is an L^p -BAPU, simultaneously for all $p \in (0, \infty]$. Thus, the decomposition spaces $\mathcal{D}_{\mathcal{F}}(\mathcal{B}, L^p, \ell_{v^{(\gamma)}}^q)$ are well-defined.

Definition 9.11. For $d \in \mathbb{N}$, $p, q \in (0, \infty]$ and $\gamma \in \mathbb{R}$, we define the **Fourier-side inhomogeneous Besov space** with integrability exponents p, q and smoothness parameter γ as

$$\mathcal{F}B_{\gamma}^{p,q}(\mathbb{R}^d) := \mathcal{F}M_{\gamma,1}^{p,q}(\mathbb{R}^d) := \mathcal{D}_{\mathcal{F}}(\mathcal{B}, L^p, \ell_{v^{(\gamma)}}^q),$$

where the weight $v^{(\gamma)} = (2^{\gamma n})_{n \in \mathbb{N}_0}$ is as in Lemma 9.10. \blacktriangleleft

As in the previous subsection, the main step for establishing embedding results is to obtain a suitable subordinateness and moderateness statement.

Lemma 9.12. Let $d \in \mathbb{N}$ and $\alpha \in [0, 1)$ be arbitrary and choose $r > r_1(d, \alpha)$, with $r_1(d, \alpha)$ as in Theorem 9.1. Then, the covering $\mathcal{Q}_r^{(\alpha)}$ is almost subordinate to \mathcal{B} .

Furthermore, we have

$$|k| \asymp \langle k \rangle \asymp 2^{n(1-\alpha)} \quad \forall k \in \mathbb{Z}^d \setminus \{0\} \text{ and } n \in \mathbb{N}_0 \text{ with } \mathcal{Q}_{r,k}^{(\alpha)} \cap B_n \neq \emptyset, \quad (9.8)$$

where the implied constants only depend on d, r, α .

In particular, $\mathcal{Q}_r^{(\alpha)}$ and $w^{(\gamma)}$ are relatively \mathcal{B} -moderate, with $w^{(\gamma)} = (\langle k \rangle^{\gamma})_{k \in \mathbb{Z}^d \setminus \{0\}}$ as in Lemma 9.2. \blacktriangleleft

Proof. Let us first show that $\mathcal{Q}_r^{(\alpha)}$ is almost subordinate to \mathcal{B} . Observe that each of the sets $\mathcal{Q}_{r,k}^{(\alpha)}$ is convex and hence path-connected and that $\mathcal{Q}_r^{(\alpha)}$ and \mathcal{B} are both admissible coverings of all of \mathbb{R}^d .

Moreover, each of the sets B_n is open, so that Corollary 2.13 implies that it suffices to show that the cardinality of the sets

$$J_k := \left\{ n \in \mathbb{N}_0 \mid B_n \cap Q_{r,k}^{(\alpha)} \neq \emptyset \right\}$$

is uniformly bounded with respect to $k \in \mathbb{Z}^d \setminus \{0\}$.

To see this, let $k \in \mathbb{Z}^d \setminus \{0\}$ be arbitrary and choose $n \in J_k$. Hence, there is some $\xi \in B_n \cap Q_{r,k}^{(\alpha)}$. Lemma 9.2 implies

$$\langle \xi \rangle \asymp \langle k \rangle^{\frac{1}{1-\alpha}}$$

where the implied constants only depend on r, α .

Let us assume $n \geq 2$ for the moment. This yields $|\xi| \geq 2^{n-2} \geq 1$ because of $\xi \in B_n$. Hence, $|\xi| \leq \langle \xi \rangle \leq 1 + |\xi| \leq 2|\xi|$. Together with $2^{n-2} < |\xi| < 2^{n+2}$, i.e. $\langle \xi \rangle \asymp |\xi| \asymp 2^n$, we arrive at

$$2^n \asymp \langle \xi \rangle \asymp \langle k \rangle^{\frac{1}{1-\alpha}} \quad \text{for } k \in \mathbb{Z}^d \setminus \{0\} \text{ and } n \in J_k \cap \mathbb{N}_{\geq 2}. \quad (9.9)$$

This yields

$$2^{\frac{\log_2 \langle k \rangle}{1-\alpha} + \log_2 C_0} = C_0 \cdot \langle k \rangle^{\frac{1}{1-\alpha}} \leq 2^n \leq C_1 \cdot \langle k \rangle^{\frac{1}{1-\alpha}} = 2^{\frac{\log_2 \langle k \rangle}{1-\alpha} + \log_2 C_1}$$

for suitable constants $C_0 \in (0, 1)$ and $C_1 \geq 1$, which only depend on r, α . We conclude

$$\log_2 C_0 \leq n - \frac{\log_2 \langle k \rangle}{1-\alpha} \leq \log_2 C_1$$

and hence $n \in \mathbb{N}_0 \cap \overline{B_R} \left(\frac{\log_2 \langle k \rangle}{1-\alpha} \right)$ for $R := \max \{-\log_2 C_0, \log_2 C_1\}$.

By dropping the assumption $n \geq 2$, we finally arrive at

$$J_k \subset \{0, 1\} \cup \left[\mathbb{N}_0 \cap \overline{B_R} \left(\frac{\log_2 \langle k \rangle}{1-\alpha} \right) \right],$$

where the latter set has cardinality at most

$$2 + 2 \lceil R \rceil + 1 = 2 \lceil R \rceil + 3 =: N.$$

As observed above, this implies that $Q_r^{(\alpha)}$ is almost subordinate to \mathcal{B} . More precisely, Lemma 2.12 shows that $Q_{r,k}^{(\alpha)} \subset B_n^{N*}$ holds for all $k \in \mathbb{Z}^d \setminus \{0\}$ and all $n \in J_k$.

Now, we establish equation (9.8). For $n \in \mathbb{N}_{\geq 2}$, and $k \in \mathbb{Z}^d \setminus \{0\}$ with $Q_{r,k}^{(\alpha)} \cap B_n \neq \emptyset$, we get $\langle k \rangle \asymp 2^{n(1-\alpha)}$ from equation (9.9). Furthermore, since $|k| \geq 1$, we also have $|k| \leq \langle k \rangle \leq 1 + |k| \leq 2|k|$ and hence $|k| \asymp \langle k \rangle \asymp 2^{n(1-\alpha)}$. This establishes equation (9.8) for $n \geq 2$. In case of $n \leq 1$, we note that Lemma 9.2 yields for arbitrary $\xi \in Q_{r,k}^{(\alpha)} \cap B_n$ that

$$1 \leq |k| \leq \langle k \rangle \asymp \langle \xi \rangle^{1-\alpha} \leq (1 + |\xi|)^{1-\alpha} < (1 + 2^{n+2})^{1-\alpha} \leq 9^{1-\alpha} \lesssim 1,$$

as well as $1 \leq 2^{n(1-\alpha)} \leq 2^{1-\alpha} \lesssim 1$, where all implied constants only depend on r and on $\alpha \in [0, 1)$. Thus, equation (9.8) holds in all cases. \square

Now, it is again easy to establish sharp embeddings between the Fourier-side α -modulation spaces and the Fourier-side inhomogeneous Besov spaces. We remark that the following theorem is essentially identical to [23, Theorem 6.2.8] from my PhD thesis.

Theorem 9.13. Let $d \in \mathbb{N}$, $\alpha \in [0, 1)$ and $p_1, p_2, q_1, q_2 \in (0, \infty]$, as well as $\gamma_1, \gamma_2 \in \mathbb{R}$.

The map

$$\iota : \mathcal{FM}_{\gamma_1, \alpha}^{p_1, q_1}(\mathbb{R}^d) \rightarrow \mathcal{FM}_{\gamma_2, 1}^{p_2, q_2}(\mathbb{R}^d), f \mapsto f$$

is well-defined and bounded if and only if we have $p_1 \leq p_2$ as well as

$$\begin{cases} \gamma_2 \leq \gamma_1 + \alpha d \left(\frac{1}{p_2} - \frac{1}{p_1} \right) + d(\alpha - 1) \left(\frac{1}{p_2'} - \frac{1}{q_1} \right)_+, & \text{if } q_1 \leq q_2, \\ \gamma_2 < \gamma_1 + \alpha d \left(\frac{1}{p_2} - \frac{1}{p_1} \right) + d(\alpha - 1) \left(\frac{1}{p_2'} - \frac{1}{q_1} \right)_+, & \text{if } q_1 > q_2. \end{cases}$$

Conversely, the map

$$\theta : \mathcal{FM}_{\gamma_1, 1}^{p_1, q_1} \rightarrow \mathcal{FM}_{\gamma_2, \alpha}^{p_2, q_2}, f \mapsto f$$

is well-defined and bounded if and only if we have $p_1 \leq p_2$, as well as

$$\begin{cases} \gamma_2 < \gamma_1 + \alpha d \left(\frac{1}{p_2} - \frac{1}{p_1} \right) + d(\alpha - 1) \left(\frac{1}{q_2} - \frac{1}{p_1^{\frac{1}{1-\alpha}}} \right)_+, & \text{if } q_1 > q_2, \\ \gamma_2 \leq \gamma_1 + \alpha d \left(\frac{1}{p_2} - \frac{1}{p_1} \right) + d(\alpha - 1) \left(\frac{1}{q_2} - \frac{1}{p_1^{\frac{1}{1-\alpha}}} \right)_+, & \text{if } q_1 \leq q_2. \end{cases}$$

Proof. Let $r_1 = r_1(d, \alpha)$ as in Theorem 9.1, choose $r > r_1$ and recall the definition of the spaces $\mathcal{FM}_{\gamma, \alpha}^{p, q}(\mathbb{R}^d) = \mathcal{D}_{\mathcal{F}}(\mathcal{Q}_r^{(\alpha)}, L^p, \ell_w^q)$ from Definition 9.4.

To characterize boundedness of ι , we define $\mathcal{Q} := \mathcal{Q}_r^{(\alpha)}$ and $\mathcal{P} := \mathcal{B}$, as well as $w := w^{(\gamma_1/(1-\alpha))}$ and $v := v^{(\gamma_2)}$, where $w^{(\gamma)}$ and $v^{(\gamma)}$ are defined as in Lemmas 9.2 and 9.10, respectively. With these choices, we have $\mathcal{FM}_{\gamma_1, \alpha}^{p_1, q_1} = \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$ and $\mathcal{FM}_{\gamma_2, 1}^{p_2, q_2} = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$.

Lemmas 9.3, 9.2 and 9.10 imply that both \mathcal{Q} and \mathcal{P} are almost structured coverings and that w and v are \mathcal{Q} -moderate and \mathcal{P} -moderate, respectively. Hence, the standing assumptions from Section 7 (i.e., Assumption 7.1) are satisfied.

Finally, \mathcal{Q}, \mathcal{P} are both coverings of the same set $\mathcal{O} = \mathcal{O}' = \mathbb{R}^d$ and Lemma 9.12 shows that \mathcal{Q} is almost subordinate to \mathcal{P} and that \mathcal{Q} and w are relatively \mathcal{P} -moderate. Thus, all assumptions of part (4) of Theorem 7.4 are satisfied. Note that the embedding ι from Theorem 7.4 satisfies $\iota f = f$ for all $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$, since both \mathcal{Q} and \mathcal{P} cover the same set $\mathcal{O} = \mathbb{R}^d = \mathcal{O}'$. Thus, the map ι from the current theorem coincides with ι from Theorem 7.4. All in all, we conclude that ι is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if

$$K := \left\| \left(\frac{v_j}{w_{i_j}} \cdot |\det T_{i_j}|^{\frac{1}{p_1} - \frac{1}{p_2} - s} \cdot |\det S_j|^s \right)_{j \in J} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} < \infty,$$

where $s = \left(\frac{1}{p_2} - \frac{1}{q_1} \right)_+$ and where for each $j \in J = \mathbb{N}_0$, some $i_j \in I_j$, i.e. with $\emptyset \neq B_j \cap \mathcal{Q}_{r, i_j}^{(\alpha)}$ is selected.

According to Lemma 9.12, this implies

$$w_{i_j} = w_{i_j}^{(\gamma_1/(1-\alpha))} = \langle i_j \rangle^{\frac{\gamma_1}{1-\alpha}} \asymp 2^{j(1-\alpha)\frac{\gamma_1}{1-\alpha}} = 2^{j\gamma_1}$$

and (with $\alpha_0 = \frac{\alpha}{1-\alpha}$)

$$|\det T_{i_j}| = |i_j|^{d\alpha_0} \asymp \left(2^{j(1-\alpha)} \right)^{d\frac{\alpha}{1-\alpha}} = 2^{jd\alpha}.$$

All in all, we get

$$\begin{aligned} K &\asymp \left\| \left(2^{j(\gamma_2 - \gamma_1)} \cdot 2^{jd\alpha(p_1^{-1} - p_2^{-1} - s)} \cdot 2^{jds} \right)_{j \in \mathbb{N}_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ &\asymp \left\| \left(2^{j(\gamma_2 - \gamma_1 + d[s(1-\alpha) + \alpha(p_1^{-1} - p_2^{-1})])} \right)_{j \in \mathbb{N}_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}. \end{aligned} \quad (9.10)$$

Recall from equation (4.3) that $q_2 \cdot (q_1/q_2)'$ is finite if and only if $q_2 < q_1$. Hence, due to the exponential nature of the sequence in equation (9.10), we get

$$\begin{aligned} K < \infty &\iff \begin{cases} \gamma_2 - \gamma_1 + d[s(1-\alpha) + \alpha(p_1^{-1} - p_2^{-1})] \leq 0, & \text{if } q_1 \leq q_2, \\ \gamma_2 - \gamma_1 + d[s(1-\alpha) + \alpha(p_1^{-1} - p_2^{-1})] < 0, & \text{if } q_1 > q_2 \end{cases} \\ &\iff \begin{cases} \gamma_2 \leq \gamma_1 + \alpha d(p_2^{-1} - p_1^{-1}) + d(\alpha - 1) \left(\frac{1}{p_2} - \frac{1}{q_1} \right)_+, & \text{if } q_1 \leq q_2, \\ \gamma_2 < \gamma_1 + \alpha d(p_2^{-1} - p_1^{-1}) + d(\alpha - 1) \left(\frac{1}{p_2} - \frac{1}{q_1} \right)_+, & \text{if } q_1 > q_2. \end{cases} \end{aligned}$$

This completes the characterization of boundedness of ι .

To characterize boundedness of θ , we define $\mathcal{Q} := \mathcal{B}$ and $\mathcal{P} := \mathcal{Q}_r^{(\alpha)}$, as well as $w := v^{(\gamma_1)}$ and $v := w^{(\gamma_2/(1-\alpha))}$, where $w^{(\gamma)}$ and $v^{(\gamma)}$ are defined as in Lemmas 9.2 and 9.10, respectively. With these choices, we have $\mathcal{FM}_{\gamma_1, 1}^{p_1, q_1} = \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$ and $\mathcal{FM}_{\gamma_2, \alpha}^{p_2, q_2} = \mathcal{D}_{\mathcal{F}}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$. Precisely as above, we see that Assumption 7.1 is fulfilled for these choices.

Finally, \mathcal{Q} and \mathcal{P} both cover the same set $\mathcal{O} = \mathbb{R}^d = \mathcal{O}'$ and Lemma 9.12 shows that \mathcal{P} is almost subordinate to \mathcal{Q} and that \mathcal{P} and v are relatively \mathcal{Q} -moderate. Thus, all assumptions of part (4)

of Theorem 7.2 are satisfied. As above, we see that the embedding ι from that theorem satisfies $\iota f = f = \theta f$ for all $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1})$ with θ as in the current theorem. Thus, all in all, we conclude that θ is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if

$$K := \left\| \left(\frac{v_{j_i}}{w_i} \cdot |\det T_i|^s \cdot |\det S_{j_i}|^{\frac{1}{p_1} - \frac{1}{p_2} - s} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'} } < \infty,$$

where $s = \left(\frac{1}{q_2} - \frac{1}{p_1 \pm \Delta} \right)_+$ and where for each $i \in I = \mathbb{N}_0$, some $j_i \in J_i$, i.e. with $\emptyset \neq B_i \cap Q_{r, j_i}^{(\alpha)}$ is selected.

With $\alpha_0 := \frac{\alpha}{1-\alpha}$ and in view of Lemma 9.12, this implies—exactly as above—that

$$v_{j_i} = \langle j_i \rangle^{\frac{\gamma_2}{1-\alpha}} \asymp 2^{i(1-\alpha)\frac{\gamma_2}{1-\alpha}} = 2^{i\gamma_2} \quad \text{and} \quad |\det S_{j_i}| = |j_i|^{d\alpha_0} \asymp 2^{i(1-\alpha)d\alpha_0} = 2^{id\alpha},$$

so that we get

$$\begin{aligned} K &\asymp \left\| \left(2^{i(\gamma_2 - \gamma_1)} \cdot 2^{ids} \cdot 2^{id\alpha(p_1^{-1} - p_2^{-1} - s)} \right)_{i \in \mathbb{N}_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'} } \\ &\asymp \left\| \left(2^{i(\gamma_2 - \gamma_1 + ds(1-\alpha) + d\alpha(p_1^{-1} - p_2^{-1}))} \right)_{i \in \mathbb{N}_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'} }. \end{aligned}$$

The remainder of the proof is exactly as above and hence omitted. \square

We note that our embedding results for α -modulation spaces from Theorem 9.7 also apply in case of $\alpha = \beta$. In contrast, the preceding theorem requires $\alpha \in [0, 1)$, so that embeddings between two (different) inhomogeneous Besov spaces are strictly speaking not covered by that criterion. This motivates the following theorem:

Theorem 9.14. Let $d \in \mathbb{N}$, $p_1, p_2, q_1, q_2 \in (0, \infty]$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ be arbitrary. The map

$$\iota : \mathcal{FM}_{\gamma_1, 1}^{p_1, q_1}(\mathbb{R}^d) \rightarrow \mathcal{FM}_{\gamma_2, 1}^{p_2, q_2}(\mathbb{R}^d), f \mapsto f$$

is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if

$$\begin{cases} \gamma_2 \leq \gamma_1 + d(p_2^{-1} - p_1^{-1}), & \text{if } q_1 \leq q_2, \\ \gamma_2 < \gamma_1 + d(p_2^{-1} - p_1^{-1}), & \text{if } q_1 > q_2. \end{cases} \quad \blacktriangleleft$$

Proof. We apply Corollary 7.3 with $\mathcal{Q} = \mathcal{B}$ and $w = v^{(\gamma_1)}$, as well as $v = v^{(\gamma_2)}$. Since \mathcal{B} is an almost structured covering by Lemma 9.10 and since the same lemma shows that w, v are \mathcal{B} -moderate, the general assumptions of Section 7 (i.e. Assumption 7.1) are satisfied.

Thus, Corollary 7.3 shows that ι is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if

$$\begin{aligned} K &:= \left\| \left(|\det T_i|^{p_1^{-1} - p_2^{-1}} \cdot \frac{v_i}{w_i} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'} } \\ &= \left\| \left(2^{nd(p_1^{-1} - p_2^{-1})} \cdot 2^{n(\gamma_2 - \gamma_1)} \right)_{n \in \mathbb{N}_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'} } \\ &= \left\| \left(2^{n[\gamma_2 - \gamma_1 + d(p_1^{-1} - p_2^{-1})]} \right)_{n \in \mathbb{N}_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'} } \end{aligned}$$

is finite. But equation (4.3) shows that $q_2 \cdot (q_1/q_2)'$ is finite if and only if we have $q_2 < q_1$. Due to the exponential nature of the weight in the preceding equation, we thus see that K is finite if and only if we have

$$\begin{cases} \gamma_2 - \gamma_1 + d(p_1^{-1} - p_2^{-1}) \leq 0, & \text{if } q_1 \leq q_2, \\ \gamma_2 - \gamma_1 + d(p_1^{-1} - p_2^{-1}) < 0, & \text{if } q_1 > q_2. \end{cases} \quad \square$$

Now, we show that our “Fourier-side” inhomogeneous Besov space indeed coincides with the Fourier transform of the usual inhomogeneous Besov space:

Lemma 9.15. Let $d \in \mathbb{N}$, $p, q \in (0, \infty]$ and $\gamma \in \mathbb{R}$ be arbitrary. Let the inhomogeneous Besov space $B_{\gamma}^{p, q}(\mathbb{R}^d) \leq \mathcal{S}'(\mathbb{R}^d)$ be defined as in²¹ [10, Definition 6.5.1].

²¹Note that the exact notation in [10, Definition 6.5.1] is actually slightly different. What we call $B_{\gamma}^{p, q}(\mathbb{R}^d)$ here is written as $B_{\gamma, q}^p$ in [10, Definition 6.5.1].

Then, the Fourier transform restricts to an isomorphism of (Quasi)-Banach spaces

$$\mathcal{F} : B_{\gamma}^{p,q}(\mathbb{R}^d) \rightarrow \mathcal{FM}_{\gamma,1}^{p,q}(\mathbb{R}^d), f \mapsto \widehat{f}|_{C_c^\infty(\mathbb{R}^d)}. \quad (9.11)$$

In particular, every $f \in \mathcal{FM}_{\gamma,1}^{p,q} \leq \mathcal{D}'(\mathbb{R}^d)$ extends to a tempered distribution. \blacktriangleleft

Proof. By [14, Lemma 2.3], there is a function $\widehat{\Psi} \in C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ with $\text{supp } \widehat{\Psi} \subset B_2(0) \setminus \overline{B_{1/2}}(0)$ and with

$$\sum_{j=-\infty}^{\infty} \widehat{\Psi}(2^{-j}\xi) = 1 \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}. \quad (9.12)$$

Let $\Psi := \mathcal{F}^{-1}\widehat{\Psi} \in \mathcal{S}(\mathbb{R}^d)$ and $\Psi_j(x) := 2^{dj} \cdot \Psi(2^j x)$ for $j \in \mathbb{Z}$ and $x \in \mathbb{R}^d$. With these definitions, we have $\widehat{\Psi_j}(\xi) = \widehat{\Psi}(2^{-j}\xi) =: \psi_j(\xi)$ for all $\xi \in \mathbb{R}^d$. From this, it follows easily that

$$\begin{aligned} \|\mathcal{F}^{-1}\psi_j\|_{L^p} &= \|\Psi_j\|_{L^p} = 2^{dj} \cdot \|\Delta_{2^j \text{id}} \Psi\|_{L^p} \\ &= 2^{dj(1-\frac{1}{p})} \|\Psi\|_{L^p} = |\det(2^j \text{id})|^{1-\frac{1}{p}} \cdot \|\Psi\|_{L^p} \end{aligned}$$

for all $j \in \mathbb{Z}$ and $p \in (0, \infty]$. Using $\text{supp } \widehat{\Psi} \subset B_2(0) \setminus \overline{B_{1/2}}(0)$, we see $\text{supp } \psi_j = 2^j \cdot \text{supp } \widehat{\Psi} \subset B_j$ for all $j \in \mathbb{N}$.

Grafakos (cf. [10, equation (6.5.2)]) states that there is a Schwartz function $\Phi \in \mathcal{S}(\mathbb{R}^d)$ satisfying

$$\widehat{\Phi}(\xi) = \begin{cases} \sum_{j \leq 0} \widehat{\Psi}(2^{-j}\xi) = \sum_{j \leq 0} \psi_j(\xi), & \text{if } \xi \neq 0, \\ 1, & \text{if } \xi = 0 \end{cases} \quad (9.13)$$

and $\widehat{\Phi}(\xi) = 1$ for $|\xi| \leq 1$, as well as $\widehat{\Phi}(\xi) = 0$ for $|\xi| \geq 2$.

Set $\varphi_i := \psi_i$ for $i \in \mathbb{N}$, and let $\varphi_0 := \widehat{\Phi}$. Using equations (9.12) and (9.13), it is not hard to see $\sum_{i \in \mathbb{N}_0} \varphi_i(\xi) = 1$ for all $\xi \in \mathbb{R}^d$. Furthermore, we have $\varphi_i \in C_c^\infty(\mathbb{R}^d)$ with $\text{supp } \varphi_i \subset B_i$ for all $i \in \mathbb{N}$ and with $\text{supp } \varphi_0 \subset \overline{B_2}(0) \subset B_4(0) = B_0$. Finally, we have

$$|\det(2^i \text{id})|^{\frac{1}{p}-1} \|\mathcal{F}^{-1}\varphi_i\|_{L^p} \leq \max \left\{ \|\Phi\|_{L^p}, \sup_{j \in \mathbb{Z}} |\det(2^j \text{id})|^{\frac{1}{p}-1} \|\mathcal{F}^{-1}\psi_j\|_{L^p} \right\} < \infty$$

for all $i \in \mathbb{N}_0$. All in all, we see that $\Phi = (\varphi_i)_{i \in \mathbb{N}_0}$ is an L^p -BAPU for \mathcal{B} for all $p \in (0, \infty]$.

Now, following Grafakos[10, Definition 6.5.1], we define

$$\Delta_j : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d), f \mapsto \mathcal{F}^{-1}(\widehat{\Psi_j} \cdot \widehat{f}) = \mathcal{F}^{-1}(\psi_j \cdot \widehat{f})$$

for $j \in \mathbb{Z}$, as well as

$$S_0 : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d), f \mapsto \Phi * f = \mathcal{F}^{-1}(\widehat{\Phi} \cdot \widehat{f}) = \mathcal{F}^{-1}(\varphi_0 \widehat{f}).$$

Using these notations, Grafakos defines the (quasi)-norm on $B_{\gamma}^{p,q}(\mathbb{R}^d)$ as

$$\begin{aligned} \|f\|_{B_{\gamma}^{p,q}} &:= \|S_0 f\|_{L^p} + \left\| (2^{j\gamma} \cdot \|\Delta_j f\|_{L^p})_{j \in \mathbb{N}} \right\|_{\ell^q} \\ &= \left\| \mathcal{F}^{-1}(\varphi_0 \cdot \widehat{f}) \right\|_{L^p} + \left\| \left(\left\| \mathcal{F}^{-1}(\varphi_j \cdot \widehat{f}) \right\|_{L^p} \right)_{j \in \mathbb{N}} \right\|_{\ell_{v(\gamma)}^q} \in [0, \infty] \end{aligned}$$

for $f \in \mathcal{S}'(\mathbb{R}^d)$. Note that we have $\widehat{f} \in \mathcal{S}'(\mathbb{R}^d) \leq \mathcal{D}'(\mathbb{R}^d)$ for $f \in \mathcal{S}'(\mathbb{R}^d)$. Now, it is not hard to see

$$\left\| \widehat{f} \right\|_{C_c^\infty(\mathbb{R}^d)} \Big|_{\mathcal{D}_{\mathcal{F}, \Phi}(\mathcal{B}, L^p, \ell_{v(\gamma)}^q)} = \left\| \left(\left\| \mathcal{F}^{-1}(\varphi_i \widehat{f}) \right\|_{L^p} \right)_{i \in \mathbb{N}_0} \right\|_{\ell_{v(\gamma)}^q} \asymp \|f\|_{B_{\gamma}^{p,q}} \quad \forall f \in \mathcal{S}'(\mathbb{R}^d), \quad (9.14)$$

so that the map \mathcal{F} defined in equation (9.11) is a well-defined isomorphism of Quasi-Banach spaces onto a closed subspace of $\mathcal{FM}_{\gamma,1}^{p,q}(\mathbb{R}^d)$.

It remains to show that the map \mathcal{F} from equation (9.11) is surjective. To this end, it suffices to show that every $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{B}, L^p, \ell_{v(\gamma)}^q)$ extends to a tempered distribution $\widetilde{f} \in \mathcal{S}'(\mathbb{R}^d)$. Indeed, if this is the case, equation (9.14) implies $\left\| \mathcal{F}^{-1} \widetilde{f} \right\|_{B_{\gamma}^{p,q}} \asymp \|f\|_{\mathcal{D}_{\mathcal{F}}(\mathcal{B}, L^p, \ell_{v(\gamma)}^q)} < \infty$ and hence $g := \mathcal{F}^{-1} \widetilde{f} \in B_{\gamma}^{p,q}(\mathbb{R}^d)$ with $\mathcal{F}g = \widetilde{f}|_{C_c^\infty(\mathbb{R}^d)} = f$, so that \mathcal{F} is surjective.

But Theorem 8.3 (with $\mathcal{Q} = \mathcal{B}$ and $I_0 = I = \mathbb{N}_0$) is precisely tailored to show that every element $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{B}, L^p, \ell_{v(\gamma)}^q)$ extends to a tempered distribution $\tilde{f} \in \mathcal{S}'(\mathbb{R}^d)$.

It is not hard to verify that the partition of unity Φ defined above is regular. Thus, all we need to verify is that there is some $N \in \mathbb{N}_0$ with $w^{(N)} \cdot c \in \ell^1(\mathbb{N}_0)$ for all $c = (c_n)_{n \in \mathbb{N}_0} \in \ell_{v(\gamma)}^q(\mathbb{N}_0)$, where

$$w_n^{(N)} = |\det T_n|^{1/p} \cdot \max \left\{ 1, \|T_n^{-1}\|^{d+1} \right\} \cdot \left[\inf_{\xi \in B_n^*} (1 + |\xi|) \right]^{-N}.$$

Because of $\ell_{v(\gamma)}^q(\mathbb{N}_0) \hookrightarrow \ell_{v(\gamma)}^\infty(\mathbb{N}_0)$, it suffices to show $w^{(N)} \cdot (v(\gamma))^{-1} \in \ell^1(\mathbb{N}_0)$ for some $N \in \mathbb{N}_0$.

Now, for $n \in \mathbb{N}$, we saw in equation (9.7) that $n^* \subset \{0\} \cup \{n-3, \dots, n+3\}$. Furthermore, directly after that equation, we showed $0^* \subset \{0, 1, 2, 3\}$, which implies $0 \notin n^*$ for $n \in \mathbb{N}_{>3}$. Thus, $n^* \subset \{n-3, \dots, n+3\}$ for $n \in \mathbb{N}_{>3}$, which easily implies $|\xi| \geq 2^{(n-3)-2} = 2^{n-5}$ for all $\xi \in B_n^*$. Furthermore, we have $\|T_n^{-1}\| = \|2^{-n} \cdot \text{id}\| = 2^{-n} \leq 1$ for all $n \in \mathbb{N}_0$, so that we get

$$w_n^{(N)} = 2^{dn/p} \cdot \left[\inf_{\xi \in B_n^*} (1 + |\xi|) \right]^{-N} \leq 2^{dn/p} \cdot (2^{n-5})^{-N} = 2^{5N} \cdot 2^{n(\frac{d}{p}-N)}$$

for all $n \in \mathbb{N}_{>3}$. Thus,

$$0 \leq \left((v(\gamma))^{-1} \cdot w^{(N)} \right)_n \leq 2^{-\gamma n} \cdot 2^{5N} \cdot 2^{n(\frac{d}{p}-N)} = 2^{5N} \cdot 2^{n(\frac{d}{p}-\gamma-N)} \quad \forall n \in \mathbb{N}_{>3},$$

which implies $(v(\gamma))^{-1} \cdot w^{(N)} \in \ell^1(\mathbb{N}_0)$, as soon as $\frac{d}{p} - \gamma - N < 0$, i.e. as soon as $N > \frac{d}{p} - \gamma$. \square

As a corollary, we can now show that the Fourier transform also yields an isomorphism between the classical “space-side” α -modulation spaces and our “Fourier-side” variants of these spaces.

Corollary 9.16. *Let $d \in \mathbb{N}$, $p, q \in (0, \infty]$, $\alpha \in [0, 1]$ and $\gamma \in \mathbb{R}$. Then, the map*

$$\mathcal{F} : M_{\gamma, \alpha}^{p, q}(\mathbb{R}^d) \rightarrow \mathcal{D}_{\mathcal{F}}\left(\mathcal{Q}^{(\alpha)}, L^p, \ell_{w^{(\gamma/(1-\alpha))}}^q\right) = \mathcal{F}M_{\gamma, \alpha}^{p, q}(\mathbb{R}^d), f \mapsto \widehat{f}|_{C_c^\infty(\mathbb{R}^d)}$$

yields an isomorphism of (Quasi)-Banach spaces, where the α -modulation space $M_{\gamma, \alpha}^{p, q}(\mathbb{R}^d)$ is defined as in [2, Definition 2.4].

In particular, our Theorems 9.7 and 9.13 concerning embeddings between the “Fourier-side” α -modulation spaces apply just as well to the classical “space-side” α -modulation spaces. \blacktriangleleft

Proof. For $\alpha = 1$, the claim was just shown in Lemma 9.15. Thus, we can assume $\alpha \in [0, 1)$.

Note that we have (by definition) that $\mathcal{F}M_{\gamma, \alpha}^{p, q}(\mathbb{R}^d) = \mathcal{D}_{\mathcal{F}}\left(\mathcal{Q}_r^{(\alpha)}, L^p, \ell_{w^{(\gamma/(1-\alpha))}}^q\right)$, for $r > r_1(d, \alpha)$, with $r_1(d, \alpha)$ as in Theorem 9.1. But this covering is exactly the same one which is used in [2, Definition 2.4] to define the space-side α -modulation spaces $M_{\gamma, \alpha}^{p, q}(\mathbb{R}^d)$. Thus, let $\Phi = (\varphi_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ be an L^p -BAPU for $\mathcal{Q}_r^{(\alpha)}$, jointly for all $p \in (0, \infty]$. Existence of Φ is ensured by Theorem 3.19, since $\mathcal{Q}_r^{(\alpha)}$ is an (almost) structured covering by Lemma 9.3. Using such a system Φ , the (quasi)-norm on $M_{\gamma, \alpha}^{p, q}(\mathbb{R}^d)$ as defined in [2, Definition 2.4], is given by

$$\|f\|_{M_{\gamma, \alpha}^{p, q}} = \left\| \left(\langle \xi_k \rangle^\gamma \cdot \left\| \mathcal{F}^{-1}(\varphi_k \widehat{f}) \right\|_{L^p} \right)_{k \in \mathbb{Z}^d \setminus \{0\}} \right\|_{\ell_q},$$

where $\xi_k \in \mathcal{Q}_{r, k}^{(\alpha)}$ can be selected arbitrarily. Note that we have $\xi_k := |k|^{\alpha_0} k \in \mathcal{Q}_{r, k}^{(\alpha)}$ and that $|\xi_k| = |k|^{\alpha_0+1} = |k|^{\frac{1}{1-\alpha}} \asymp \langle k \rangle^{\frac{1}{1-\alpha}}$ (since $k \in \mathbb{Z}^d \setminus \{0\}$), so that we get $\langle \xi_k \rangle^\gamma \asymp \langle k \rangle^{\gamma/(1-\alpha)} \asymp w_k^{(\gamma/(1-\alpha))}$. All in all, it is now not hard to see that we have

$$\|f\|_{M_{\gamma, \alpha}^{p, q}} \asymp \left\| \left(\left\| \mathcal{F}^{-1}(\varphi_k \widehat{f}) \right\|_{L^p} \right)_{k \in \mathbb{Z}^d \setminus \{0\}} \right\|_{\ell_{w^{(\gamma)}}^q} = \left\| \widehat{f} \right\|_{C_c^\infty(\mathbb{R}^d)} \Big|_{\mathcal{D}_{\mathcal{F}}(\mathcal{Q}_r^{(\alpha)}, L^p, \ell_{w^{(\gamma/(1-\alpha))}}^q)} \quad \forall f \in \mathcal{S}'(\mathbb{R}^d), \quad (9.15)$$

which shows that the map \mathcal{F} defined above is a well-defined isomorphism of Quasi-Banach spaces onto a closed subspace of $\mathcal{D}_{\mathcal{F}}\left(\mathcal{Q}_r^{(\alpha)}, L^p, \ell_{w^{(\gamma/(1-\alpha))}}^q\right)$.

Thus, to complete the proof, we only have to show that \mathcal{F} is surjective. To this end, it suffices to show that each distribution $f \in \mathcal{D}_{\mathcal{F}} \left(\mathcal{Q}_r^{(\alpha)}, L^p, \ell_{w(\gamma)}^q \right) = \mathcal{FM}_{\gamma, \alpha}^{p, q}(\mathbb{R}^d)$ can be extended to a *tempered* distribution $\tilde{f} \in \mathcal{S}'(\mathbb{R}^d)$. Indeed, if this is the case, equation (9.15) shows

$$\left\| \mathcal{F}^{-1} \tilde{f} \right\|_{M_{\gamma, \alpha}^{p, q}} \asymp \left\| \tilde{f} \right\|_{C_c^\infty(\mathbb{R}^d)} \Big|_{\mathcal{FM}_{\gamma, \alpha}^{p, q}} = \|f\|_{\mathcal{FM}_{\gamma, \alpha}^{p, q}} < \infty$$

and hence $g := \mathcal{F}^{-1} \tilde{f} \in M_{\gamma, \alpha}^{p, q}(\mathbb{R}^d)$ with $\mathcal{F}g = \tilde{f}|_{C_c^\infty(\mathbb{R}^d)} = f$, so that \mathcal{F} is surjective.

But Theorem 9.13 shows that we have $\mathcal{FM}_{\gamma, \alpha}^{p, q}(\mathbb{R}^d) \hookrightarrow \mathcal{FM}_{\sigma, 1}^{p, q}(\mathbb{R}^d)$ for a suitable $\sigma \in \mathbb{R}$, so that every $f \in \mathcal{FM}_{\gamma, \alpha}^{p, q}(\mathbb{R}^d)$ also satisfies $f \in \mathcal{FM}_{\sigma, 1}^{p, q}(\mathbb{R}^d)$. But in view of Lemma 9.15, this implies that f extends to a tempered distribution, as desired. \square

9.3. Embeddings between homogeneous and inhomogeneous Besov spaces. In the previous subsections, the coverings which we considered were very *compatible*: They all covered the same set (namely, \mathbb{R}^d) and \mathcal{Q} was almost subordinate to \mathcal{P} and relatively \mathcal{P} -moderate, or vice versa. Thus, we could completely characterize existence of the associated embeddings, but we could not indicate in how far our theory also applies in cases where the two coverings are more incompatible to each other.

Hence, in this subsection, we consider embeddings between inhomogeneous and homogeneous (Fourier-side) Besov spaces. The inhomogeneous Besov covering was introduced in Definition 9.9; it consists of dyadic annuli, but the low-frequency part is covered by one large ball. In contrast, the homogeneous Besov covering only consists of dyadic annuli:

Definition 9.17. For $d \in \mathbb{N}$ and $n \in \mathbb{Z}$, define $\dot{B}_n := B_{2^{n+2}}(0) \setminus \overline{B_{2^{n-2}}}(0) \subset \mathbb{R}^d$. We define the **homogeneous Besov covering** of \mathbb{R}^d as $\dot{\mathcal{B}} := (\dot{B}_n)_{n \in \mathbb{Z}}$. \blacktriangleleft

As in the previous subsections, our first aim is to show that $\dot{\mathcal{B}}$ is an almost structured covering. In this case, $\dot{\mathcal{B}}$ is in fact structured:

Lemma 9.18. Let $d \in \mathbb{N}$. Then $\dot{\mathcal{B}} = (T_n Q + b_n)_{n \in \mathbb{Z}}$ is a structured admissible covering of $\mathbb{R}^d \setminus \{0\}$, with standardizations given by

$$T_n := 2^n \cdot \text{id}, \quad b_n := 0, \quad \text{and} \quad Q := B_4(0) \setminus \overline{B_{1/4}}(0)$$

for $n \in \mathbb{Z}$.

Furthermore, for $\gamma \in \mathbb{R}$, the weight $w^{(\gamma)} := (2^{n\gamma})_{n \in \mathbb{Z}}$ is $\dot{\mathcal{B}}$ -moderate. \blacktriangleleft

Proof. With T_n, b_n, Q as in the statement of the lemma, we clearly have $\dot{B}_n = T_n Q + b_n$ for all $n \in \mathbb{Z}$.

Now, let $P := B_2(0) \setminus \overline{B_{1/2}}(0)$. It is clear that $P, Q \subset \mathbb{R}^d$ are open and bounded, with $\overline{P} \subset Q$. Furthermore,

$$\mathcal{P} := (P_n)_{n \in \mathbb{Z}} := (T_n P + b_n)_{n \in \mathbb{Z}} = (B_{2^{n+1}}(0) \setminus \overline{B_{2^{n-1}}}(0))_{n \in \mathbb{Z}}$$

is a covering of $\mathbb{R}^d \setminus \{0\}$. Indeed, clearly $P_n \subset \mathbb{R}^d \setminus \{0\}$ for all $n \in \mathbb{Z}$. Conversely, for arbitrary $\xi \in \mathbb{R}^d \setminus \{0\}$, we can choose $n \in \mathbb{Z}$ maximal with $2^n \leq |\xi|$, since the set of those $n \in \mathbb{Z}$ is nonempty (this uses $|\xi| > 0$ and $2^n \xrightarrow{n \rightarrow -\infty} 0$), closed and bounded from above. Then $2^{n-1} < 2^n \leq |\xi| < 2^{n+1}$, which means $\xi \in P_n$.

Since we have $P_n \subset \dot{B}_n \subset \mathbb{R}^d \setminus \{0\}$ for all $n \in \mathbb{Z}$, we see that $\dot{\mathcal{B}}$ is also a covering of $\mathbb{R}^d \setminus \{0\}$. To show that $\dot{\mathcal{B}}$ is a structured admissible covering of $\mathbb{R}^d \setminus \{0\}$, it remains to show that $\dot{\mathcal{B}}$ is admissible and that

$$C_{\dot{\mathcal{B}}} := \sup_{n \in \mathbb{Z}} \sup_{k \in n^*} \|T_n^{-1} T_k\| = \sup_{n \in \mathbb{Z}} \sup_{k \in n^*} 2^{k-n} \quad (9.16)$$

is finite. But for $n \in \mathbb{Z}$ and $k \in n^*$, there is some $\xi \in \dot{B}_k \cap \dot{B}_n$, which implies

$$2^{n-2} < |\xi| < 2^{k+2} \quad \text{and} \quad 2^{k-2} < |\xi| < 2^{n+2}.$$

Hence, $n-2 < k+2$ and $k-2 < n+2$, which implies $n-4 < k < n+4$. Since all of these quantities are integers, we conclude $k \in \{n-3, \dots, n+3\}$ and thus

$$n^* \subset \{n-3, \dots, n+3\}, \quad (9.17)$$

which yields $|n^*| \leq 7$. Since this holds for all $n \in \mathbb{Z}$, $\dot{\mathcal{B}}$ is admissible.

But we also get $2^{k-n} \leq 2^{(n+3)-n} = 2^3 = 8$. Thanks to equation (9.16), this yields $C_{\dot{\mathcal{B}}} \leq 8 < \infty$, as desired.

Finally, we also get $\dot{\mathcal{B}}$ -moderateness of $w^{(\gamma)}$: For $n \in \mathbb{Z}$ and $k \in n^*$, we saw above that $|k - n| \leq 3$. Hence,

$$\left| \frac{w_k^{(\gamma)}}{w_n^{(\gamma)}} \right| = 2^{\gamma(k-n)} \leq 2^{|\gamma(k-n)|} \leq 2^{3|\gamma|},$$

so that we get $C_{w^{(\gamma)}, \dot{\mathcal{B}}} \leq 2^{3|\gamma|} < \infty$. \square

In view of the preceding lemma, the following decomposition spaces are well-defined:

Definition 9.19. For $d \in \mathbb{N}$, $p, q \in (0, \infty]$ and $\gamma \in \mathbb{R}$, we define the **homogeneous (Fourier-side) Besov space** with exponents p, q and smoothness γ as

$$\mathcal{F}\dot{B}_{\gamma}^{p,q}(\mathbb{R}^d) := \mathcal{D}_{\mathcal{F}}(\dot{\mathcal{B}}, L^p, \ell_{w^{(\gamma)}}^q) \leq \mathcal{D}'(\mathbb{R}^d \setminus \{0\}),$$

with $w^{(\gamma)} = (2^{n\gamma})_{n \in \mathbb{Z}}$. \blacktriangleleft

Our next step is to verify that the homogeneous Besov covering $\dot{\mathcal{B}}$ is almost subordinate to the inhomogeneous Besov covering \mathcal{B} .

Lemma 9.20. Let $d \in \mathbb{N}$ be arbitrary. $\dot{\mathcal{B}}$ is almost subordinate to \mathcal{B} and we have

$$\{n\} \subset \left\{ k \in \mathbb{Z} \mid \dot{B}_k \cap B_n \neq \emptyset \right\} \subset \{n-3, \dots, n+3\} \quad (9.18)$$

for $n \in \mathbb{N}$ and

$$\mathbb{Z}_{\leq 1} \subset \left\{ k \in \mathbb{Z} \mid \dot{B}_k \cap B_0 \neq \emptyset \right\} \subset \mathbb{Z}_{\leq 3}. \quad (9.19) \quad \blacktriangleleft$$

Proof. For $k \in \mathbb{N}$, we simply have $\dot{B}_k = B_k = B_k^{0*}$. But for $k \in \mathbb{Z}_{\leq 0}$, we have

$$\dot{B}_k = B_{2^{k+2}}(0) \setminus \overline{B_{2^{k-2}}}(0) \subset B_{2^2}(0) = B_4(0) = B_0 = B_0^{0*}.$$

All in all, we have shown that $\dot{\mathcal{B}}$ is almost subordinate to \mathcal{B} with $k(\dot{\mathcal{B}}, \mathcal{B}) = 0$.

Now, for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ with $\emptyset \neq \dot{B}_k \cap B_n = \dot{B}_k \cap \dot{B}_n$, equation (9.17) implies $k \in \{n-3, \dots, n+3\}$, which establishes equation (9.18).

Finally, for $k \in \mathbb{Z}$ with $\dot{B}_k \cap B_0 \neq \emptyset$, there is some $\xi \in B_0 \cap \dot{B}_k$, which implies $2^{k-2} < |\xi| < 4 = 2^2$ and hence $k < 4$, i.e. $k \leq 3$, which establishes the right inclusion in equation (9.19).

Conversely, for $k \in \mathbb{Z}_{\leq 1}$, we have $\xi := 2^k \cdot (1, 0, \dots, 0) \in \dot{B}_k$ and $|\xi| \leq 2^1 < 4$, which implies $\xi \in B_0 \cap \dot{B}_k \neq \emptyset$. This establishes the left inclusion in equation (9.19). \square

Now, we are in a position to consider embeddings between the homogeneous and the inhomogeneous (Fourier-side) Besov spaces. We remark that the following theorem already appeared in my PhD thesis, [23, Theorem 6.2.6].

Theorem 9.21. Let $d \in \mathbb{N}$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in (0, \infty]$. Then, the map

$$\iota : \mathcal{F}B_{\gamma_1}^{p_1, q_1}(\mathbb{R}^d) \rightarrow \mathcal{F}\dot{B}_{\gamma_2}^{p_2, q_2}(\mathbb{R}^d), f \mapsto f|_{C_c^\infty(\mathbb{R}^d \setminus \{0\})}$$

is well-defined and bounded if we have $p_1 \leq p_2$ and

$$\begin{cases} \gamma_2 \leq \gamma_1 + d(p_2^{-1} - p_1^{-1}), & \text{if } q_1 \leq q_2, \\ \gamma_2 < \gamma_1 + d(p_1^{-1} - p_2^{-1}), & \text{if } q_1 > q_2, \end{cases} \quad (9.20)$$

as well as

$$\begin{cases} \gamma_2 \geq d(p_2^{-1} - \min\{1, p_1^{-1}\}), & \text{if } p_1^\Delta \leq q_2, \\ \gamma_2 > d(p_2^{-1} - \min\{1, p_1^{-1}\}), & \text{if } p_1^\Delta > q_2. \end{cases} \quad (9.21)$$

Conversely, if ι is bounded, then we have $p_1 \leq p_2$, condition (9.20) is fulfilled and we have

$$\begin{cases} \gamma_2 \geq d(p_2^{-1} - p_1^{-1}), & \text{if } p_1 \leq q_2, \\ \gamma_2 > d(p_2^{-1} - p_1^{-1}), & \text{if } p_1 > q_2. \end{cases} \quad (9.22)$$

Furthermore, we also have

$$\begin{cases} \gamma_2 \geq d(p_2^{-1} - 1), & \text{if } q_2 = \infty, \\ \gamma_2 > d(p_2^{-1} - 1), & \text{if } q_2 < \infty. \end{cases} \quad (9.23)$$

Finally, in case of $p_1 = p_2$, we also have

$$\begin{cases} \gamma_2 \geq 0, & \text{if } 2 \leq q_2, \\ \gamma_2 > 0, & \text{if } 2 > q_2. \end{cases} \quad (9.24) \quad \blacktriangleleft$$

Remark. Note that the conditions $p_1 \leq p_2$ and (9.20) belong to the sufficient conditions as well as to the necessary conditions derived above.

Thus, the only difference between the sufficient and the necessary conditions is that condition (9.21) is not fully equivalent to the conjunction of conditions (9.22) and (9.23). There are, however, many cases in which the equivalence *does* hold: For $p_1 \in [2, \infty]$, we have $p_1^\Delta = p_1$ and $\min\{1, p_1^{-1}\} = p_1^{-1}$, so that conditions (9.21) and (9.22) are equivalent. Furthermore, for $p_1 \in (0, 1]$, we have $p_1^\Delta = \infty$ and $\min\{1, p_1^{-1}\} = 1$, so that conditions (9.21) and (9.23) are equivalent.

Thus, the only case in which we do *not* get a complete characterization is if we have

$$p_1 \leq p_2, \quad p_1 \in (1, 2), \quad \gamma_2 = d(p_2^{-1} - p_1^{-1}) \quad \text{and finally} \quad p_1 \leq q_2 < p_1'. \quad (9.25)$$

In this case, the necessary conditions are fulfilled, but the sufficient conditions are not, so that our criteria cannot decide whether ι is or is not bounded.

In case of $p_1 = p_2 =: p$, this “gap” between necessary and sufficient criteria can be narrowed further. In this case, condition (9.24) shows that the only case in which we do *not* get a complete characterization is if we have

$$p \in (1, 2), \quad \gamma_2 = 0 \quad \text{and finally} \quad 2 \leq q_2 < p_1'. \quad (9.26)$$

If conditions (9.25) or (9.26) are fulfilled, our criteria are inconclusive. But I do not know of any more comprehensive results in this direction which could clarify this ambiguity. In fact, the only statement about embeddings of inhomogeneous into homogeneous Besov spaces of which I know is [22, Theorem in §2.3.3], which shows that

$$f \mapsto \|f\|_{L^p} + \|f\|_{\dot{B}_\gamma^{p,q}}$$

is an equivalent norm on the space $B_\gamma^{p,q}(\mathbb{R}^d)$, provided that $\gamma > \sigma_p = d\left(\frac{1}{p} - 1\right)_+$. In particular, this yields $B_\gamma^{p,q}(\mathbb{R}^d) \hookrightarrow \dot{B}_\gamma^{p,q}(\mathbb{R}^d)$ for $\gamma > d\left(\frac{1}{p} - 1\right)_+$. Note, however, that our results from above show that this embedding is even true for $\gamma \geq d\left(\frac{1}{p} - 1\right)_+$, as long as $q \geq p^\Delta$. In this sense, our results improve that given by Triebel in [22].

I would be thrilled about any further progress in closing the gap described in equation (9.25). \blacklozenge

Proof. Define

$$\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i)_{i \in I} := \mathcal{B} \quad \text{and} \quad \mathcal{P} = (P_j)_{j \in J} = (S_j P)_{j \in J} := \dot{\mathcal{B}},$$

as well as $w := v^{(\gamma_1)} = (2^{n\gamma_1})_{n \in \mathbb{N}_0}$ and $v := w^{(\gamma_2)} = (2^{k\gamma_2})_{k \in \mathbb{Z}}$. Here, we have $T_i = 2^i \text{id}$ and $S_j = 2^j \text{id}$. In view of Lemmas 9.10 and 9.18, we see that \mathcal{Q} and \mathcal{P} are both almost structured coverings of $\mathcal{O} := \mathbb{R}^d$ and $\mathcal{O}' := \mathbb{R}^d \setminus \{0\}$, respectively. Furthermore, the same lemmas also show that w and v are \mathcal{Q} -moderate and \mathcal{P} -moderate, respectively, so that the standing assumptions of Section 7 (i.e., Assumption 7.1) are satisfied.

Finally, thanks to Lemma 9.20, we know that \mathcal{P} is almost subordinate to \mathcal{Q} (but *not* relatively \mathcal{Q} -moderate). For the sufficient condition, we can thus apply part (1) of Theorem 7.2 to conclude that ι is well-defined and bounded if we have $p_1 \leq p_2$ and if $K_{p_1^\Delta, 2} < \infty$ (in the notation of Theorem 7.2).

To verify this, we more generally investigate finiteness of

$$\begin{aligned} K(r, \alpha, \beta) &:= \left\| \left(w_i^{-1} \cdot |\det T_i|^\alpha \cdot \left\| \left(|\det S_j|^\beta \cdot v_j \right)_{j \in J_i} \right\|_{\ell^{q_2 \cdot (r/q_2)'}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ &= \left\| \left(2^{n d \alpha} \cdot 2^{-n \gamma_1} \cdot \left\| (2^{k d \beta} \cdot 2^{k \gamma_2})_{k \in \mathbb{Z}^{(n)}} \right\|_{\ell^{q_2 \cdot (r/q_2)'}} \right)_{n \in \mathbb{N}_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \end{aligned}$$

for arbitrary $r \in (0, \infty]$ and $\alpha, \beta \in \mathbb{R}$ and with

$$\mathbb{Z}^{(n)} := \left\{ k \in \mathbb{Z} \mid \dot{B}_k \cap B_n \neq \emptyset \right\}. \quad (9.27)$$

We first observe that $K(r, \alpha, \beta)$ is finite if and only if we have

$$K_0(r, \alpha, \beta) := \left\| (2^{k d \beta} \cdot 2^{k \gamma_2})_{k \in \mathbb{Z}^{(0)}} \right\|_{\ell^{q_2 \cdot (r/q_2)'}} < \infty$$

and if

$$\begin{aligned} \tilde{K}(r, \alpha, \beta) &:= \left\| \left(2^{n(d\alpha - \gamma_1)} \cdot \left\| (2^{k(\gamma_2 + d\beta)})_{k \in \mathbb{Z}^{(n)}} \right\|_{\ell^{q_2 \cdot (r/q_2)'}} \right)_{n \in \mathbb{N}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ (\text{equation (9.18)}) &\asymp \left\| \left(2^{n(d\alpha - \gamma_1)} \cdot 2^{n(\gamma_2 + d\beta)} \right)_{n \in \mathbb{N}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ &= \left\| \left(2^{n(\gamma_2 - \gamma_1 + d(\alpha + \beta))} \right)_{n \in \mathbb{N}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \end{aligned}$$

is finite. Since equation (4.3) shows that $q_2 \cdot (q_1/q_2)'$ is finite if and only if $q_1 > q_2$, we see that $\tilde{K}(r, \alpha, \beta)$ is finite if and only if we have

$$\begin{cases} \gamma_2 - \gamma_1 + d(\alpha + \beta) \leq 0, & \text{if } q_1 \leq q_2, \\ \gamma_2 - \gamma_1 + d(\alpha + \beta) < 0, & \text{if } q_1 > q_2. \end{cases} \quad (9.28)$$

Furthermore, equation (9.19) from Lemma 9.20 yields $\mathbb{Z}_{\leq 1} \subset \mathbb{Z}^{(0)} \subset \mathbb{Z}_{\leq 3}$. Additionally, by another application of equation (4.3), we see that $q_2 \cdot (r/q_2)'$ is finite if and only if $r > q_2$, so that $K_0(r, \alpha, \beta)$ is finite if and only if we have $\left\| (2^{k(\gamma_2 + d\beta)})_{k \in \mathbb{Z}_{\leq 1}} \right\|_{\ell^{q_2 \cdot (r/q_2)'}} < \infty$ and thus if and only if

$$\begin{cases} \gamma_2 + d\beta \geq 0, & \text{if } r \leq q_2, \\ \gamma_2 + d\beta > 0, & \text{if } r > q_2. \end{cases} \quad (9.29)$$

All in all, we see that $K(r, \alpha, \beta)$ is finite if and only if conditions (9.28) and (9.29) are satisfied.

Now, there are two cases. In case of $p_1 \geq 1$, the quantity $K_{p_1^\Delta, 2}$ from Theorem 7.2 satisfies $K_{p_1^\Delta, 2} = K_{p_1^\Delta, 1} = K\left(p_1^\Delta, 0, \frac{1}{p_1} - \frac{1}{p_2}\right)$. Plugging these values into conditions (9.28) and (9.29), we see that these conditions are valid, thanks to our assumptions (9.20) and (9.21). To see this, note that we have $\min\{1, p_1^{-1}\} = p_1^{-1}$, since $p_1 \geq 1$.

In case of $p_1 < 1$, the quantity $K_{p_1^\Delta, 2}$ from Theorem 7.2 satisfies $K_{p_1^\Delta, 2} = K\left(p_1^\Delta, \frac{1}{p_1} - 1, 1 - \frac{1}{p_2}\right)$. Plugging these values into conditions (9.28) and (9.29), we again see that these conditions are valid, thanks to our assumptions (9.20) and (9.21). To see this, note in this case that $\min\{1, p_1^{-1}\} = 1$, since $p_1 \in (0, 1)$.

Now, let us derive the stated necessary conditions. To this end, assume that ι is bounded. This easily implies boundedness of the map θ from part (2) of Theorem 7.2, since ι maps each function $f \in C_c^\infty(\mathcal{O}') = C_c^\infty(\mathbb{R}^d \setminus \{0\})$ to itself (interpreted as a distribution on $\mathbb{R}^d \setminus \{0\}$). Hence, part (2) of Theorem 7.2 yields $p_1 \leq p_2$ and shows that we have $K\left(p_1, 0, \frac{1}{p_1} - \frac{1}{p_2}\right) = K_{p_1, 1} < \infty$. In view of conditions (9.28) and (9.29), this shows that conditions (9.20) and (9.22) from the statement of the theorem are indeed fulfilled.

Next, if we have $p_1 = p_2$, we can apply part (3) of Theorem 7.2, which shows that $K_{2,1} = K(2, 0, 0)$ is finite. In view of condition (9.29), we thus see that condition (9.24) from the statement of the theorem is indeed fulfilled.

It remains to show that condition (9.23) is also fulfilled. This appears to be difficult, since we have exhausted the necessary criteria provided by Theorem 7.2. But in this case, it helps to invoke Lemma 6.19, with our choices $\mathcal{Q} = \mathcal{B}$ and $\mathcal{P} = \dot{\mathcal{B}}$ from above and with $J_0 = J = \mathbb{Z}$. Indeed, as required in Lemma 6.19, \mathcal{P}_{J_0} is almost subordinate to \mathcal{Q} .

Furthermore, Lemma 6.19 requires existence of a bounded map ι (which we will write as ι_0 from now on to avoid confusion with the map ι from the current theorem) which satisfies $\langle \iota_0 f, \varphi \rangle = \langle f, \varphi \rangle$ for all $\varphi \in C_c^\infty(\mathcal{O} \cap \mathcal{O}') = C_c^\infty(\mathbb{R}^d \setminus \{0\})$ and all $f \in \mathcal{D}_K^{\mathcal{Q}, p_1, \ell_w^{q_1}}$, where K is some subset of $\mathcal{O} = \mathbb{R}^d$ (defined in Lemma 6.19), whose precise definition is immaterial for us.

In our present case, we can simply select $\iota_0 := \iota|_{\mathcal{D}_K^{\mathcal{Q}, p_1, \ell_w^{q_1}}}$. Indeed, for $f \in \mathcal{D}_K^{\mathcal{Q}, p_1, \ell_w^{q_1}} \subset C_c^\infty(\mathbb{R}^d)$ (interpreted as a distribution on \mathbb{R}^d) and $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$, we have

$$\langle \iota f, \varphi \rangle = \langle f|_{C_c^\infty(\mathbb{R}^d \setminus \{0\})}, \varphi \rangle = \langle f, \varphi \rangle,$$

as required. Now, since we have $\delta_0 \in \ell_w^{q_1}(\mathbb{N}_0)$, Lemma 6.19 shows (for $i = 0$) that we have

$$\begin{aligned} \infty > u_i &= |\det T_i|^{\frac{1}{p_1}-1} \cdot \left\| \left(v_j \cdot |\det S_j|^{1-\frac{1}{p_2}} \right)_{j \in J_0 \cap J_i} \right\|_{\ell^{q_2}} \\ &= \left(\text{since } i = 0, \text{ since } J_0 = \mathbb{Z} \text{ and since } J_i = \mathbb{Z}^{(i)} \right) = \left\| \left(2^{j\gamma_2} \cdot 2^{jd(1-\frac{1}{p_2})} \right)_{j \in \mathbb{Z}^{(0)}} \right\|_{\ell^{q_2}}, \end{aligned}$$

with $\mathbb{Z}^{(0)}$ as in equation (9.27). But because of $\mathbb{Z}_{\leq 1} \subset \mathbb{Z}^{(0)} \subset \mathbb{Z}_{\leq 3}$, we see that u_0 is finite if and only if condition (9.23) is satisfied. \square

As our next result, we study the “reverse” direction of the preceding theorem, i.e. embeddings of the homogeneous into the inhomogeneous Besov spaces. We remark that the following theorem already appeared in a slightly different form in my PhD thesis, as [23, Theorem 6.2.3].

Theorem 9.22. Let $d \in \mathbb{N}$, $p_1, p_2, q_1, q_2 \in (0, \infty]$ and $\gamma_1, \gamma_2 \in \mathbb{R}$.

If we have $p_1 \leq p_2$,

$$\begin{cases} \gamma_2 \leq \gamma_1 + d(p_2^{-1} - p_1^{-1}), & \text{if } q_1 \leq q_2, \\ \gamma_2 < \gamma_1 + d(p_2^{-1} - p_1^{-1}), & \text{if } q_1 > q_2. \end{cases} \quad (9.30)$$

and

$$\begin{cases} \gamma_1 \leq d(p_1^{-1} - p_2^{-1}), & \text{if } q_1 \leq p_2^\nabla, \\ \gamma_1 < d(p_1^{-1} - p_2^{-1}), & \text{if } q_1 > p_2^\nabla. \end{cases} \quad (9.31)$$

then there is a bounded linear map

$$\iota : \mathcal{FB}_{\gamma_1}^{p_1, q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{FB}_{\gamma_2}^{p_2, q_2}(\mathbb{R}^d)$$

satisfying

$$\langle \iota f, \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\}) \text{ and } f \in \mathcal{FB}_{\gamma_1}^{p_1, q_1}(\mathbb{R}^d)$$

and $\iota f = f$ for all $f \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$.

Conversely, if a map with these properties exist, then we necessarily have $p_1 \leq p_2$, condition (9.30) is satisfied and we have

$$\begin{cases} \gamma_1 \leq d(p_1^{-1} - p_2^{-1}), & \text{if } q_1 \leq p_2 < \infty, \\ \gamma_1 < d(p_1^{-1} - p_2^{-1}), & \text{if } q_1 > p_2, \\ \gamma_1 < d(p_1^{-1} - p_2^{-1}), & \text{if } p_2 = \infty \text{ and } q_1 > 1 = p_2^\nabla, \\ \gamma_1 \leq d(p_1^{-1} - p_2^{-1}), & \text{if } p_2 = \infty \text{ and } q_1 \leq 1 = p_2^\nabla. \end{cases} \quad (9.32)$$

Finally, in case of $p_1 = p_2$, we also have

$$\begin{cases} \gamma_1 \leq 0, & \text{if } q_1 \leq 2, \\ \gamma_1 < 0, & \text{if } q_1 > 2. \end{cases} \quad (9.33) \quad \blacktriangleleft$$

Remark. As for the previous theorem, we precisely determine the case in which our criteria are inconclusive. The condition $p_1 \leq p_2$ and condition (9.30) appear in both the necessary and the sufficient conditions. Hence, the only difference is between conditions (9.31) and (9.32). In case of $p_2 \in (0, 2]$, we have $p_2^\vee = p_2$, so that both conditions coincide. They also coincide for $p_2 = \infty$. Thus, the *only* case in which our criteria are inconclusive is if

$$p_1 \leq p_2, \quad p_2 \in (2, \infty), \quad \gamma_1 = d(p_1^{-1} - p_2^{-1}), \quad \text{and finally} \quad p_2' < q_1 \leq p_2. \quad (9.34)$$

In this case, the necessary conditions are fulfilled, but the sufficient conditions are not.

In case of $p_1 = p_2 =: p$, we can narrow this region down even further, using condition (9.33). In this case, the only case in which our criteria are inconclusive is if

$$p \in (2, \infty), \quad \gamma = 0 \quad \text{and finally} \quad p_2' < q_1 \leq 2. \quad (9.35)$$

Finally, note that our necessary conditions *always* entail $\gamma_1 \leq d(p_1^{-1} - p_2^{-1})$ and

$$\gamma_2 \leq \gamma_1 + d(p_2^{-1} - p_1^{-1}) \leq d(p_1^{-1} - p_2^{-1}) + d(p_2^{-1} - p_1^{-1}) = 0,$$

so that the homogeneous Besov space can only ever embed into inhomogeneous Besov spaces of *non-positive* smoothness. \blacklozenge

Proof. Define

$$\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q)_{i \in I} = \dot{\mathcal{B}} \quad \text{and} \quad \mathcal{P} = (P_j)_{j \in J} = (S_j P_j')_{j \in J} = \mathcal{B},$$

as well as $w := w^{(\gamma_1)} = (2^{k\gamma_1})_{k \in \mathbb{Z}}$ and $v := v^{(\gamma_2)} = (v^{n\gamma_2})_{n \in \mathbb{N}_0}$. Here, we have $T_i = 2^i \text{id}$ and $S_j = 2^j \text{id}$. In view of Lemmas 9.10 and 9.18, we see that \mathcal{Q} and \mathcal{P} are both almost structured coverings of $\mathcal{O} := \mathbb{R}^d \setminus \{0\}$ and $\mathcal{O}' := \mathbb{R}^d$, respectively. Furthermore, the same lemmas also show that w and v are \mathcal{Q} -moderate and \mathcal{P} -moderate, respectively, so that the standing assumptions of Section 7 (i.e., Assumption 7.1) are satisfied.

Finally, thanks to Lemma 9.20, we know that \mathcal{Q} is almost subordinate to \mathcal{P} (but *not* relatively \mathcal{P} -moderate). For the sufficient condition, we can thus apply part (1) of Theorem 7.4 to conclude that a map ι with properties as in the current theorem exists if we have $p_1 \leq p_2$ and if $K_{p_2^\vee} < \infty$ (in the notation of Theorem 7.4). To see this, note that the map ι given by Theorem 7.4 satisfies that ιf is an extension of $f \in \mathcal{D}'(\mathcal{O}) = \mathcal{D}'(\mathbb{R}^d \setminus \{0\})$ to $C_c^\infty(\mathbb{R}^d)$, for each $f \in \mathcal{D}_{\mathcal{F}}(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}) = \mathcal{F}\dot{B}_{\gamma_1}^{p_1, q_1}(\mathbb{R}^d)$, so that we get $\langle \iota f, \varphi \rangle = \langle f, \varphi \rangle$ for all $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$. Furthermore, the map ι from Theorem 7.4 also fulfills $\iota f = f$ for all $f \in C_c^\infty(\mathcal{O}) = C_c^\infty(\mathbb{R}^d \setminus \{0\})$, as desired.

To verify $K_{p_2^\vee} < \infty$, we more generally investigate finiteness of

$$\begin{aligned} K(r) &:= K_r = \left\| \left(v_j \cdot \left\| \left(|\det T_i|^{\frac{1}{p_1} - \frac{1}{p_2}} / w_i \right)_{i \in I_j} \right\|_{\ell^{r \cdot (q_1/r)'} } \right)_{j \in J} \right\|_{\ell^{q_2 \cdot (q_1/q_2)' } } \\ &= \left\| \left(2^{n\gamma_2} \cdot \left\| \left(2^{kd(p_1^{-1} - p_2^{-1})} / 2^{k\gamma_1} \right)_{k \in \mathbb{Z}^{(n)}} \right\|_{\ell^{r \cdot (q_1/r)'} } \right)_{n \in \mathbb{N}_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)' } } \end{aligned}$$

for general $r \in (0, \infty]$, where

$$\mathbb{Z}^{(n)} := \left\{ k \in \mathbb{Z} \mid \dot{B}_k \cap B_n \neq \emptyset \right\}$$

is defined as in equation (9.27). Recall from Lemma 9.20 that we have $\{n\} \subset \mathbb{Z}^{(n)} \subset \{n-3, \dots, n+3\}$ for $n \in \mathbb{N}$ and $\mathbb{Z}_{\leq 1} \subset \mathbb{Z}^{(0)} \subset \mathbb{Z}_{\leq 3}$.

Now, we observe that $K(r)$ is finite if and only if

$$K_0(r) := \left\| \left(2^{kd(p_1^{-1} - p_2^{-1})} / 2^{k\gamma_1} \right)_{k \in \mathbb{Z}^{(0)}} \right\|_{\ell^{r \cdot (q_1/r)'} } = \left\| \left(2^{k[-\gamma_1 + d(p_1^{-1} - p_2^{-1})]} \right)_{k \in \mathbb{Z}^{(0)}} \right\|_{\ell^{r \cdot (q_1/r)'} }$$

is finite and if furthermore

$$\begin{aligned}
 \tilde{K}(r) &:= \left\| \left(2^{n\gamma_2} \cdot \left\| \left(2^{kd(p_1^{-1}-p_2^{-1})/2^{k\gamma_1}} \right)_{k \in \mathbb{Z}(n)} \right\|_{\ell^{r \cdot (q_1/r)'} } \right)_{n \in \mathbb{N}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\
 (\text{equation (9.18)}) &\asymp \left\| \left(2^{n\gamma_2} \cdot 2^{nd(p_1^{-1}-p_2^{-1})/2^{n\gamma_1}} \right)_{n \in \mathbb{N}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\
 &= \left\| \left(2^{n[\gamma_2 - \gamma_1 + d(p_1^{-1}-p_2^{-1})]} \right)_{n \in \mathbb{N}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}
 \end{aligned}$$

is finite. But equation (4.3) shows that the exponent $q_2 \cdot (q_1/q_2)'$ is finite if and only if $q_2 < q_1$. Hence, due to the exponential nature of the sequence above, we conclude

$$\tilde{K}(r) < \infty \iff \begin{cases} \gamma_2 - \gamma_1 + d(p_1^{-1} - p_2^{-1}) \leq 0, & \text{if } q_1 \leq q_2, \\ \gamma_2 - \gamma_1 + d(p_1^{-1} - p_2^{-1}) < 0, & \text{if } q_1 > q_2. \end{cases} \quad (9.36)$$

Note that the right-hand side is actually independent of r .

Finally, recalling $\mathbb{Z}_{\leq 1} \subset \mathbb{Z}^{(0)} \subset \mathbb{Z}_{\leq 3}$ and noting that $r \cdot (q_1/r)'$ is finite if and only if $r < q_1$, we also conclude

$$K_0(r) < \infty \iff \begin{cases} -\gamma_1 + d(p_1^{-1} - p_2^{-1}) \geq 0, & \text{if } q_1 \leq r, \\ -\gamma_1 + d(p_1^{-1} - p_2^{-1}) > 0, & \text{if } q_1 > r. \end{cases} \quad (9.37)$$

If we now recall from above that a map ι with the properties claimed in the current theorem exists if we have $p_1 \leq p_2$ and if $K_{p_2^\nabla} < \infty$, we see that the sufficient conditions stated in the current theorem are indeed sufficient for existence of ι .

Now, we consider the necessary conditions. To this end, assume that a map ι as in the statement of the current theorem exists. Since ι satisfies $\iota f = f$ for all $f \in C_c^\infty(\mathbb{R}^d \setminus \{0\}) = C_c^\infty(\mathcal{O})$, we see that the map θ from part (2) of Theorem 7.4 is well-defined and bounded. Hence, that part of Theorem 7.4 shows that we have $p_1 \leq p_2$ and $K_s < \infty$, with $s = p_2$ for $p_2 \in (0, \infty)$ and with $s = 1 = p_2^\nabla$ for $p_2 = \infty$. In view of conditions (9.36) and (9.37), we thus see that conditions (9.30) and (9.32) from the statement of the current theorem are indeed fulfilled.

Finally, if we have $p_1 = p_2$, we can apply part (3) of Theorem 7.4, which yields $K_2 < \infty$. Just as above, this implies that condition (9.33) is indeed fulfilled. \square

Before we close this section, we remark that one can show—similar to Lemma 9.15—that the map

$$\dot{B}_{\gamma}^{p,q}(\mathbb{R}^d) \leq \mathcal{S}'(\mathbb{R}^d) / \mathcal{P} \rightarrow \mathcal{D}_{\mathcal{F}}(\dot{B}, L^p, \ell_{v(\gamma)}^q) = \mathcal{F}\dot{B}_{\gamma}^{p,q}(\mathbb{R}^d), f + \mathcal{P} \mapsto \hat{f}|_{C_c^\infty(\mathbb{R}^d \setminus \{0\})} \quad (9.38)$$

is an isomorphism of (Quasi)-Banach spaces, cf. [23, Lemma 6.2.2]. This justifies our notation $\mathcal{F}\dot{B}_{\gamma}^{p,q}(\mathbb{R}^d)$. Furthermore, using this isomorphism, it is easy to see that the embedding results from Theorems 9.21 and 9.22 also yield corresponding embedding results for the “usual, space-side” homogeneous and inhomogeneous Besov spaces.

The main step in showing that (9.38) indeed yields an isomorphism is to show that each distribution $f \in \mathcal{D}_{\mathcal{F}}(\dot{B}, L^p, \ell_{v(\gamma)}^q) \leq \mathcal{D}'(\mathbb{R}^d \setminus \{0\})$ can be extended to a *tempered* distribution $\tilde{f} \in \mathcal{S}'(\mathbb{R}^d)$. To prove this, however, we cannot use Theorem 8.3, since this theorem relies on the quantity

$$c_i := \inf_{\xi \in Q_i^*} (1 + |\xi|)$$

being large for most $i \in I$. But for the homogeneous Besov covering, we have $c_i \asymp 1$ for all $i \in \mathbb{Z}_{\leq 0}$. The actual proof given in [23, Lemma 6.2.2] is quite technical and does not yield a further illustration of our embedding results. Hence, we omit the proof.

CONCLUDING REMARKS

In this paper, we developed a comprehensive and easy to use framework for embeddings between decomposition spaces. This framework essentially allows to reduce the study of embeddings between the Fourier analytic decomposition spaces to *purely combinatorial conditions*. To check whether these conditions are satisfied, *no knowledge of Fourier analysis is required* anymore. Instead, all one needs to

understand is the relation between the two coverings under consideration and the associated discrete sequence spaces.

The applications given in Section 9 underline this claim: Once we understood the relation between the different α -coverings, or between the homogeneous and the inhomogeneous Besov coverings, application of the criteria produced by our embedding framework was straightforward.

Note, however, that we were unable to *completely* characterize the existence of embeddings between homogeneous and inhomogeneous Besov spaces. This shows that for highly “incompatible” coverings, the developed theory is still incomplete—although it is more comprehensive than any previously known results. The desire to close this gap provides an interesting topic for further research.

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